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CONTENTS

		Page
Extension of Telephone Plant with Regard to the Value of Subscribers' Time	Y Rapp	3
A Theory of the Eddy Current Equivalent Winding and its Application to the Closing of Non-Delayed Telephone Relays. A Study of Telephone Relays (4)	S Ekelöf	103
Frequency Components in the Output of a Harmonic Generator Driven by the Sum of Two Sine Waves	G Einarsson	139

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Extension of Telephone Plant with Regard to the Value of Subscribers' Time

I. Economic Optimum for Conductor Diameter and Number of Switches on Telephone Circuits

BY

YNGVE RAPP*

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This paper presents an attempt to arrive at an objective aid for guidance in decisions concerning

the number of switches on a route and the corresponding traffic losses

the diameter of conductors in local telephone cables and the corresponding reference equivalent for a telephone circuit within a national network.

These questions will be dealt with on economic principles starting from the premise that, in deciding how much money to spend on a telephone plant, the subscribers' demands for accessibility and intelligibility should be taken into account by weighing the cost of conversations against the cost to subscribers in the form of lost time due to traffic losses and to unsatisfactory transmission. This principle implies that the plant must be so constructed and dimensioned that the sum of the plant cost and of the monetary value of the inconvenience to subscribers from traffic losses and unsatisfactory transmission is a minimum. The investigation involves a system analysis, the problem being to find the combination of different components providing the economically optimum result under given assumptions regarding the engineering standard.

The study is based on the premise that the quantities which describe the requirement, viz. the number of subscribers and the frequency function for the traffic, undergo no change during the period under consideration.

The question of the appropriate time and the appropriate capacity for the various stages of expansion to cater for a still growing need will be dealt with in a separate paper in this issue.

In a third paper to be published later, the question of the reference equivalent for telephone circuits will be examined in greater detail.

Paper to be presented at The Third International Teletraffic Congress in Paris, September 11—16, 1961.

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* Telefonaktiebolaget L M Ericsson, Stockholm.

Contents

Page

CHAPTER 1 *Introduction* 5

CHAPTER 2 *Definition of Traffic, Traffic Losses and Number of Subscribers* 6

CHAPTER 3 *Fundamental Theory for Determining the Reference Equivalent and Conductor Diameter in Telephone Networks and the Traffic Losses and Number of Switches on a Traffic Route* 9

3.1 Outside plant cost 11

3.2 Exchange equipment cost 11

3.3 Inconvenience to subscribers 11

3.4 Economic optimum for conductor diameter and number of switches. . 21

CHAPTER 4 *Different Methods of Estimating theQuality of Service Factor G Representing the Value of Subscribers' Time* 23

4.1 Time evaluation 23

4.2 Maximization of profits 24

4.3 Substitution 24

4.4 Plant cost 25

4.5 As-if principle and reconciliation of the hypotheses 27

4.6 Quality of service factor G in long distance traffic 28

CHAPTER 5 *The Inconveniences Attendant on Use of the Telephone* 29

ANNEX 1 *Simplification of the Problem of Determining Cable Conductor Diameter and Number of Switches by Considering It as Two Separate Problems* . . 34

ANNEX 2 *Dependence of Result on Estimate of Quality of Service Factor, Traffic and Rate of Interest* 36

ANNEX 3 *Calculation of the Number of Switches on a Traffic Route with a Single Switching Stage. Discussion on Basis of a Numerical Example* 44

ANNEX 4 *Calculation of Number of Switches on a Route Comprising Several Selector Stages* 51

ACKNOWLEDGEMENTS 58

BIBLIOGRAPHY 59

CHAPTER 1

Introduction

The object of a telephone plant is to establish connection between two persons without loss of time and to transmit correct messages on the circuit. The opinion of a subscriber on the quality of connections is based essentially on the degree to which these two requirements are fulfilled.

Among the considerations on which a subscriber bases his opinion of his telephone, the following are the most important:

1. How quickly can connection be established?
2. To what degree and how quickly can one make oneself understood in conversation with different subscribers?
3. What do the conversations cost?

Among other considerations which affect the subscriber's view of the quality of telephone connections may be mentioned the fault rate, how quickly faults are repaired, and the facilities offered by the administration in respect of special services such as directory enquiries, absent subscriber's service, etc. In this paper account will be taken solely of points 1—3 above. The important question of the most economic maintenance effort will thus not be dealt with.

Savings can be made in telephone plant by reducing the number of switches and trunks. But this leads to high congestion, with the consequent inconvenience to subscribers.

Savings can likewise be made by reducing the diameter of conductors in the cables. But this results in reduced intelligibility of communication, leading to misunderstanding and repeated questioning, again with inconvenience to subscribers.

On the other hand investments in telephone exchanges and cables in order to guarantee the quickest possible connection and perfect intelligibility under all circumstances cannot be increased indefinitely, for the cost of conversation would then exceed its usefulness to the subscriber and the value he places upon it.

Therefore, when deciding on the amount of money to spend on telephone plant to satisfy the subscribers' demands in respect of accessibility of switching paths and quality of transmission, one should consider the cost of conversation against the cost to subscribers of the inconvenience which must inevitably arise, and choose a size of plant for which the cost of the plant and the economic value of the inconvenience to subscribers together form a minimum.

An investigation of this kind must be organized in different ways and yield different results according to whether it is concerned with ideal conditions, allowing full freedom

of choice of the optimal combination of plant components, or is aimed at the coordination of different systems for incorporation of a given system into the whole. In both cases, however, certain assumptions must be made concerning the properties and the cost structure of the systems employed. In brief, the investigation must be based on a given system or on a coordinated group of systems.

Thus, if different systems are to be compared on a cost basis, it is necessary to establish for each of them the optimal combination of plant components. The total cost of each system should, of course, include the switching cost characteristic of each. In order to be able to calculate the optimal combination of plant elements, the relevant costs must be expressible as a function of certain plant provision parameters such as the number of switches in an exchange or the diameter of conductors in a cable. Furthermore, in order to match the stages of plant expansion both to the anticipated requirement and to the losses of subscribers' time, the calculation must be based on certain measures which define this requirement.

CHAPTER 2

Definition of Traffic, Traffic Losses and Number of Subscribers

An indispensable condition for arriving at an optimization on the principles outlined above is that the inconveniences to subscribers in terms of traffic losses and inaudibility of transmitted speech can be evaluated in monetary terms. Among the various imaginable means of evaluating the quality of the service, the author has chosen to introduce a factor G having the nature of an irreducible social constant which may be said to represent the value of subscribers' time. Some methods which may be used as a guide in estimating this constant are suggested in chapter 4; and the degree to which calculations are inevitably affected by the uncertainty of the estimate is discussed in Annex 2, where it is found that the dependence lies solely on the magnitude of G.

In addition to the factor G, describing the inconvenience to subscribers, we need certain measures for traffic, traffic losses and number of subscribers; we must also draw upon certain considerations deriving from traffic theory. We shall now devote some attention to these questions.

In contradistinction to many other forms of activity, there is no difficulty in telephony, at least in principle, in finding suitable measures which satisfactorily define the magnitude of the need. These measures are the flow of traffic and the number of subscribers, which, when sufficiently specified with regard to the project in view, provide a satisfactory basis for determining the extent of plant provision and the most appropriate times for the various stages of expansion.

A plant extension is the result of a decision; and once the decision has been made, some time is required for installation of the plant, which may vary from a month or so for switches in a telephone exchange to several years for the installation of new exchanges and cable plant. As already indicated, the actual decision function contains two elements, a time element and a quantity element, so that every future extension of plant requires a decision as to when the preparations must start and what capacity to plan for. The decision naturally implies, too, a choice between different ways and means of supplying the expected need. We shall assume in the sequel that the question of method has been decided or can be decided by repeated alternative comparative cost estimates.

Behind a decision for plant extension there are always certain ideas about the measures — traffic and number of subscribers — which indicate the future requirement.

As is known, the traffic through different switches and routes varies from hour to hour and from day to day during the year. Therefore it cannot very well be defined on the basis of a mean value or of the traffic during a number of busy hours, but its variations during the entire year must be taken into account. Under these circumstances the traffic at a given point in time and during a given period, such as a year, is defined by a statistical distribution, the frequency function of which is called $f(A)$. If the traffic grows from year to year, this frequency curve changes character, for example as shown in *Fig. 2.1*, being displaced more and more to the right and with high traffic values becoming increasingly common.

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Fig. 2.1. Frequency function $f(A)$ for the traffic at different times $t = 0, 1, 2$. The $t = 0$ curve is taken from a paper by Anders Elldin and Ingvar Tånge, *Long-time observations of telephone traffic*, Tele No 1, 1960, English ed. Curves $t = 0$, $t = 1$, $t = 2$ describe traffic quantities, the mean intensities of which have the ratios 1 : 1.25 : 1.50 and are calculated from the identity

$$\int_0^\infty f(A) \cdot dA = \int_0^\infty \frac{1}{K} \cdot f\left(\frac{A}{K}\right) \cdot dA = 1$$

For $t = 0, 1, 2$, then $K = 1, 1.25, 1.50$.

The frequency function for a future annual period cannot be assessed with full assurance, and it is therefore advisable to form an ensemble of frequency functions for the period and allot to each a certain probability $P_t[f(A)]$.

In the same way the number of future subscribers, N, is defined by a probability $P_t(N)$.

At a given traffic, A, the probability of congestion, E, is dependent on the number of switches $n_1, n_2 \ldots$ in the various switching stages, *i.e.* $E = E(n_1, n_2 \ldots A)$, which for the sake of simplicity is written $E = E(n, A)$. Then $A \cdot E(n, A)$ represents the traffic which may be expected to encounter congestion at an initiated flow of traffic A.

On the assumption that the frequency function $f(A)$ describes the traffic during one year, which means that there is a traffic A during the time $f(A) \cdot dA$ years, the quantity of traffic disturbed during one year will be[1, 2, 3].

$$\int_0^\infty A \cdot E(n, A) \cdot f(A) \cdot dA \text{ erlang years} \tag{2.1}$$

Since the contributions to the quantity of traffic disturbed during the entire year, determined by means of the above expression, are insignificant as soon as the traffic is below and the number of switches above a given limit, the traffic disturbed during low traffic conditions may be neglected. It may therefore be convenient to let $f(A)$ represent only the periods for which the traffic exceeds a given value. Assume by way of example that $f(A)$ instead represents, on the average, T hours per day. The amount of traffic disturbed during a year can then obviously be written

$$365 \cdot T \int_0^\infty A \cdot E(n, A) \cdot f(A) \cdot dA \text{ erlang hours} \tag{2.2}$$

The above definitions of traffic, disturbed quantity of traffic, and number of subscribers, form one of the bases used in this paper for discussing the dispositions which should be taken to meet the future requirement.

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1 Cf. *Moe's Principle. An econometric investigation intended as an aid in dimensioning and managing telephone plant*, by A Jensen, Copenhagen 1950, p. 34.

2 Cf. *Traffic theory as an aid in the planning of and operation of the road grid*, by A Jensen, Ingeniøren, International Edition, No. 2, Vol. 1.

3 Cf. *Long-time observations of telephone traffic*, by Anders Elldin and Ingvar Tånge, Tele, No. 1, 1960, English ed.

CHAPTER 3

Fundamental Theory for Determining the Reference Equivalent and Conductor Diameter in Telephone Networks and the Traffic Losses and Number of Switches on a Traffic Route

The theory here presented for provision of telephone plant with regard to accessibility and intelligibility is based exclusively on an evaluation of the inconvenience to subscribers from traffic losses in switches and from unsatisfactory transmission on circuits.

With the guidance of this evaluation it is shown how the number of switches on traffic routes and the reference equivalent of a circuit shall be determined so that the sum of the plant costs and of the economic value of the inconvenience to subscribers attain a minimum.

To carry out this calculation, the following cost items must be separately analysed so as to be calculable and addable under different assumptions as to the number of switches on traffic routes and the dimensions of conductors in the network:

1. Cost of subscribers' cables	K_a
2. Cost of exchanges	K_s
3. Value of inconvenience to subscribers due to	
a) unsatisfactory transmission	K_{t_1}
b) traffic losses	K_{t_2}

Therefore it should be possible to establish these costs as functions of the number of switches $n_1, n_2 \ldots$ and of the conductor diameters $z_1, z_2 \ldots$, the which magnitudes must simultaneously be determined so that the total cost

$$K(n_1, n_2 \ldots\ldots z_1, z_2 \ldots\ldots) = K_a + K_s + K_{t_1} + K_{t_2}$$

is as small as possible.

The inconvenience to subscribers on the score of unintelligibility is measured in terms of the prolongation of conversation time occurring for a given quantity of traffic and given conductor dimensions.

Inconvenience to subscribers in respect of inaccessibility is measured in terms of the proportion of traffic disturbance with a given number of switches, based on a frequency function $f(A)$ for the traffic during the entire year.

These inconveniences are converted into monetary terms by using an irreducible social constant G representing the value of subscribers' time.

Present values are used throughout. For example, a cost of b kronor per switch includes all charges for operation, maintenance and plant replacement. This is achieved by multiplying the first cost b' by a present value factor μ (see Annex 2), so that

$$b = \mu \cdot b'$$

From the foregoing it is apparent that existing standards of grade of service and recommendations concerning the reference equivalent of a transmission system, concepts such as busy hour and certain traditional measures of intelligibility, have no place in this theory, which makes no presuppositions in these respects.

The question of determining the capacity of a traffic route on economic principles was first taken up by MOE[1] and shortly thereafter in precise mathematical form by ERLANG[2].

These ideas were later developed by A JENSEN[3] into an economic theory for determining the number of switches on traffic routes. He shows how to calculate the requirement of switches for as low as possible a cost consistent with maximum predetermined values of total congestion.

An analogous method was used by the present author[4] for determining the conductor diameter in subscribers' lines and junction cables consistent with maximum sending and receiving reference equivalents as recommended by C.C.I.T.T.

In the present study the problem has been expanded and an attempt is made to present a theory aimed at simultaneous determination of the number of switches and of reference equivalents for telephone circuits.

In order to arrive at a simple and clear presentation, without loss of general validity, we may consider an isolated telephone exchange with its subscribers' network. The problem may be formulated as follows.

Determine the conductor diameter in the subscribers' cables and the number of switches in the exchange for a given number of subscribers and an initiated traffic of given frequency function $f(A)$ so that the sum of the cost of subscribers' cables and exchange equipment and of the value of inconvenience to subscribers is as small as possible.

It is assumed that this telephone plant is built of a number of components of a standard type and that the cost of cable plant and exchange equipment can be calculated if the quantities of different components are known. To study how the total cost changes with traffic loss and transmission level, it must be broken down among the three cost centres — outside plant, exchange equipment, and the value of inconvenience to subscribers.

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[1] MOE: Til Driftsingeniøren. Angående økonomikurver. (Jan. 1924).
MOE: Til Driftsingeniøren. Referat af en afhandling af F. Spiecker: *Die Abhängigkeit des erfolgreichen Fernsprechanrufes von der Anzahl der Verbindungsorgane.* (April 1925).
MOE: I henhold til direktørens ønske om at få »Busy« gjort up to date fremsendes . . . (Oct. 1927).

[2] Transactions of the Danish Academy of Technical Sciences, A.T.S. 1948, No. 2, *The Life and Works of A K Erlang* by E BROCKMEYER, H L HALSTRØM and A JENSEN, pp. 216—221.

[3] *Moe's Principle. An econometric investigation intended as an aid in dimensioning and managing telephone plant. Theory and tables*, by A JENSEN, Copenhagen 1950.

[4] Y RAPP: *The economic optimum in urban telephone network problems*, Ericsson Technics 1950, pp. 52—71.

3.1 Outside plant cost

The cost of a cable plant depends on a number of different factors such as the subscribers' geographical distribution, the number of subscribers, and the conductor dimensions. Of these three factors the conductor dimension alone is of interest in this context, since we are considering a particular plant and it is only the conductor dimension which affects the transmission level. The outside plant cost may therefore be written

$$K_a = K_{0a} + c \cdot l_m z^2 \cdot N \qquad \text{kronor}^1 \tag{3.1}$$

where

K_{0a} = costs independent of conductor diameter, kr
c = a constant, kr/km mm^2
l_m = mean length of line in the network, km
z = diameter of conductor in the cables, mm
N = number of subscribers

3.2 Exchange equipment cost

In the exchange a connection passes a number of switches which operate both in parallel and in series and whose number can be matched to the quantity of traffic. The cost of an exchange, at least within certain limits as regards the number of switches, can be split up as follows:

$$K_s = a + \sum_i b_i \cdot n_i \qquad \text{kronor} \tag{3.2}$$

where

a = a constant cost independent of the number of switches
b_i = the cost of a switch in switching group i
n_i = the number of switches in switching group i

3.3 Inconvenience to subscribers

Under unsatisfactory conditions of transmission and congestion in switches, certain inconveniences arise for the subscribers, in the form partly of pure losses of time and partly of risks. With unsatisfactory transmission there is a risk of misunderstanding. High congestion involves the risk of a message not being communicated at the right time, which might lead to a financial loss.

3.3.1 Inconvenience due to unsatisfactory transmission

Unsatisfactory transmission impairs intelligibility, and the conversation time is prolonged through the need for repetition. Furthermore, the recipient may read a wrong meaning into a message, and misunderstandings arise which cannot be put right.

Time losses due to unsatisfactory transmission represent a cost which is assumed to be directly proportional to the prolongation of the conversation time.

---

[1] Kronor, abbreviated kr, stands for Swedish Crowns, the unit of Swedish currency (1 U.S. $ ≃ 5 kr).

A misunderstanding can give rise to undesirable consequences involving financial loss.

Accordingly, inconvenience to subscribers due to unsatisfactory transmission may be represented by the sum of time costs and of risk costs.

$$K_{t_1} = [2G \cdot T_0 \cdot (\vartheta_m - 1) + \varrho] \cdot N \text{ kronor} \tag{3.3}$$

where

G = an irreducible social constant by which the inconvenience to subscribers in the form of prolonged conversation time is converted into monetary terms. This constant may be said to represent the capitalized value of one hour per day of the subscriber's time. If, for example, the subscriber's time is worth 8 kr. per hour and the rate of interest is 8 per cent per annum

$$G = \frac{8 \cdot 365}{0.08} = 36{,}500 \text{ kr.} \quad \text{(see chap. 4.)}$$

T_0 = expected total duration of calls in hours initiated per subscriber per d ay under idea transmission conditions

ϑ_m = prolongation factor, *i.e.* the mean time of actual conversation in relation to T_0

ϱ = capitalized value per subscriber of the risk of misunderstanding

N = number of subscribers (as before)

Both ϑ_m and ϱ may be assumed to be non-diminishing functions of the overall reference equivalent, R, of the telephone circuit, so long at least as the value of R does not fall below a given limit R_* and does not exceed a value above which intelligibility is seriously hampered.

One way of estimating the prolongation factor is to determine the time required for a person fully to comprehend a particular message. The principle has been used for determining the course of the function $\vartheta = f(R)$.

Tests conducted by H. Hansson, and the statistical analysis thereof made by G. Lind, indicate that the result of the tests can conveniently be described by the formula[1]

$$\vartheta = e^{\gamma (R - R_*)^2} \qquad R > R_*$$

$$\vartheta = 1 \qquad R \leq R_*$$

where

γ = a constant with the expected value $\gamma = 7.34 \times 10^{-4}$

R_* = upper limit for the overall reference equivalent R below which no prolongation of the time of conversation due to lack of intelligibility is expected. This limit was estimated at $R = 30$ db.

Fig. 3.1 shows the course of this function.

---

[1] A description of the tests conducted and of their statistical analysis will be given in a separate paper, *Estimation of the Increase in Conversation Time as Function of the Overall Reference Equivalent of a Telephone Circuit* by H. Hansson and G. Lind, to be published in a later issue.

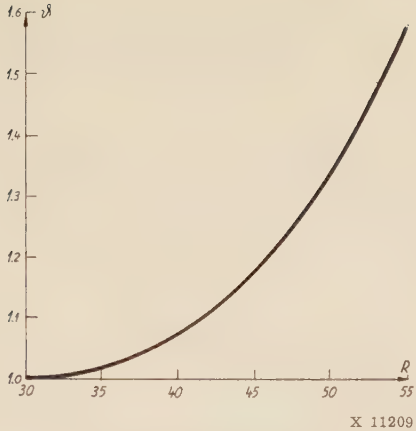

Fig. 3.1. The prolongation factor
$\vartheta = e^{7.34 \cdot 10^{-4}(R-30)^2}$
as function of the overall reference equivalent R.

An estimate of the risk of misunderstanding might be based on the change in intelligibility with transmission level.

But there would appear to be considerable difficulty in evaluating these risks in terms of money. In all oral communication there is some risk of misunderstanding, which is independent of transmission level but dependent solely on the ability of the parties to express themselves and to understand one another.

For this reason if for no other, it is difficult to separate the portion due to transmission level alone. The latter portion, moreover, should undergo comparatively slight variations, at least within the limits of transmission level to which the calculations lead.

Moreover, since neither the prolongation factor (ϑ_m) nor the "quality of service factor" (G) can be accurately estimated, and the result is dependent primarily on the magnitude of the service factor[1], it appears best to include the risk of misunderstanding in the latter.

The mean prolongation factor ϑ_m is determined from the following expression, which holds good provided that the community of interest between different subscribers is constant over the entire area under consideration:

$$\vartheta_m = \iint f(y) \cdot f(x) \cdot \vartheta_{yx} \cdot dy \cdot dx \tag{3.4}$$

[1] See Annex 2.

where

$f(u)$ = frequency function for geographical distribution of subscribers (u = distance from the exchange, km)

ϑ_{yx} = $f(R_{yx})$ or the prolongation factor in a message from a subscriber at distance y km from the exchange to a subscriber at distance x km from the exchange at a reference equivalent, R_{yx}, between the subscribers ($R_{yx} \neq R_{xy}$)

Here

$$R_{yx} = R_0 + (\alpha + \alpha_m) \cdot y + \alpha \cdot x \text{ db}$$

R_0 = reference equivalent independent of the conductor diameter in the network

$$\alpha = \frac{c_1}{z} = \text{a. c. attenuation, db/km}$$

$$\alpha_m = \frac{c_2}{z^2} = \text{transmitter feed attenuation, db/km}$$

c_1, c_2 = constants

z = conductor diameter in mm

When calculating eq. 3.4 it must be noted that the prolongation factor becomes unity as soon as the reference equivalent R_{yx} falls below a given value R_*. The expression is therefore best written as follows, which gives the relative prolongation of conversation time:

$$\vartheta_m - 1 = \iint_{\Omega} f(y) \cdot f(x) \cdot (\vartheta_{yx} - 1) \cdot dy \cdot dx \tag{3.4 a}$$

with the integration domain

$$\Omega: \begin{cases} (\alpha + \alpha_m) \cdot y + \alpha \cdot x + R_0 \geq R_* \\ x \leq L, \qquad y \leq L \end{cases} \tag{3.5}$$

where

L = the longest line in the network, km.

The above expression, which is true of a single network or of two identical networks, can be easily generalized to apply to networks with different frequency functions for the geographical distribution of the subscribers and with different geographical areas, as discussed in the paper *The Reference Equivalent within a National Network* to be published in a later issue.

3.3.2 Inconvenience due to congestion

On a call from a subscriber A to a subscriber B, connection is not always established. The following are the chief causes:

1. Congestion in switching circuit in a lost call system. The subscriber chooses not to wait in a delay system.

2. Subscriber B engaged on another call.
3. Subscriber B absent or fails to answer for other reason.
4. Technical fault on the circuit or fault of operator.
5. Wrong dialling by subscriber.

Of all these causes, the first constitutes a very small fraction. The probability of congestion or of a subscriber being kept waiting is usually only about one-twentieth of other causes.

If, for any of the reasons listed, subscriber A fails to obtain connection with subscriber B, he is faced with the choice either of abstaining from the connection or of making one or more renewed attempts.

To guard against the possibility of a wrong connection, he perhaps makes one new attempt. Whether or not he thereafter makes further attempts will depend partly on his knowledge of the other person's habits in respect of availability for conversation, and partly on the value he attaches to the connection. After some time, and after a number of renewed attempts, the value of the connection in his eyes may fall so that he no longer considers it worth the trouble and expense of going on trying. From this premise, and having regard to subscribers' traffic requirements, certain repeated calling habits have developed which, together with the traffic requirement, make up the traffic initiated by subscribers. As we have already seen, this can be represented by a frequency function $f(A)$.

If, in a delay system, subscribers were altogether to abstain from waiting when they encounter congestion the delay system would function as a lost call system.

In a lost call system, on the other hand, if subscribers were to repeat their calls until they obtain connection, the likelihood of congestion would be practically the same as if all blocked calls waited until they were handled.

Depending on the patience of subscribers, on their ability to wait, and on the number of renewed attempts in a delay system, and on the frequency of renewed attempts in a lost call system, the delay system to some extent assumes the character of a lost call system and a lost call system to some extent the character of a delay system; and, according to the nature of the traffic, the real congestion, P, must lie at some point between the congestion E in a pure lost call system and congestion D in a pure delay system, so that

$$E < P < D$$

We shall now give closer thought to the conditions in a lost call system in which repeated calls occur[1, 2, 3]. In a lost call system a figure of more than 50 per cent repetitions of congested

[1] Cf. Anders Elldin and Ingvar Tånge: *Long-time observations of telephone traffic*, Tele No. 1, 1960, English ed., p. 17, fig. 4.10.

[2] Cf. Charles Clos: *An aspect of the dialing behaviour of subscribers and its effect on the trunk plant*, The Bell System Technical Journal, Vol. XXVII, July 1948, No. 3, pp. 424—445.

[3] Cf. J. W. Cohen: *Basic problems of telephone traffic theory and the influence of repeated calls*, Philips Telecommunication Review, Vol. 18, No. 2, August 1957.

calls is probably quite common, at least if the subscribers do not have to pay for attempted calls but only for established connections. Repeated calls cause subscribers certain losses of time, representing a cost which must be taken into account in the switch computation. The assertion that the time spent by a subscriber on repeated calling is of no value leads immediately to the conclusion that the number of switches must be as large or a little larger than the subscribers' traffic requirements, since an unlimited or at all events large number of repeated calls might achieve the desired connection.

These losses of time are made up of the time spent by the subscriber in replacing the handset after an unsuccessful attempt, reaching the telephone again, raising the handset and dialling the number. These time losses represent a constant cost per repeated call irrespective of the traffic. The number of repeated calls, on the other hand, obviously depends on the accessibility, Q, of the system:

$$Q = [1 - E(n, A)]\,(1 - H) \tag{3.6}$$

where

$E(n, A)$ = the congestion in switches

H = failure of connection for other reasons (points 2—5 above)

With full accessibility, *i.e.* $Q = 1$, no repeated calls are needed; but as accessibility diminishes, the number of repeated calls grows, at least as long as Q does not fall to a point at which subscribers no longer consider it worth while to make the attempt. But an increase in the number of repeated calls causes an increase in the initiated traffic which, for a constant number of switches, lowers the accessibility and may increase the number of repeated calls. Finally, it has been found in practice that an increase in accessibility may in itself lead to a greater need of traffic.

It will be apparent that the subscribers' traffic requirement, accessibility, repeated calls and initiated traffic are mutually dependent. This dependence is illustrated in *Fig. 3.2* both graphically and in a greatly simplified analytical form which is intended merely to indicate the nature of the relationships. From this figure one may imagine that a state of equilibrium is successively built up by the following chain process.

The subscribers' need of a volume of traffic M_0 initiates, to start with, a volume of traffic M_0 which, with a given number of switches, n, gives the congestion E.

The reduced accessibility, Q, due to congestion or failure to obtain connection on other grounds causes the subscribers to make a certain number of repeated calls which add a quantity S to the volume of traffic. For the same reason the subscribers' traffic requirement diminishes from M_0 to M, since a number of not immediately obtainable connections have lost in importance. The result will be either an increase or decrease in the initiated traffic A.

An increase of A reduces Q; S increases and M diminishes. This chain process continues until the increase in S is as great as the reduction of M, and at a state of equilibrium the condition must apply that

$$A = M + S \tag{3.7}$$

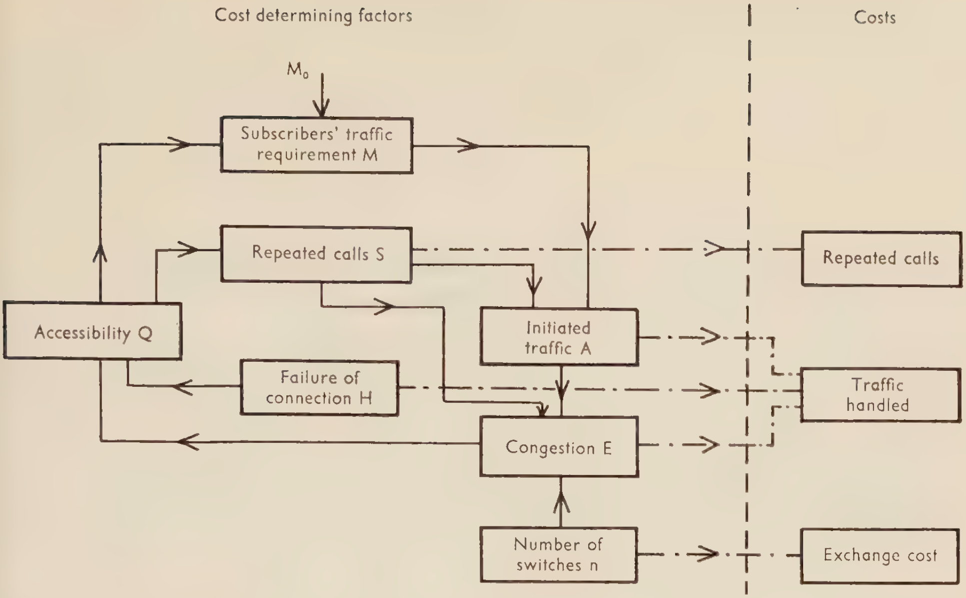

Fig. 3.2. If E increases, M diminishes and S increases. Within a given interval $E_1 < E < E_2$, M and S can be developed in series of powers of E.

$$M = (1 - \eta)A_0\,[1 - q_1 E + \ldots] \qquad S = \eta \cdot A_0\,[1 + q_2 E + \ldots.]$$

where A_0 and η = constants

$$A = M + S = A_0(1 - qE + \ldots\ldots) \qquad q = (1 - \eta)\,q_1 - \eta q_2$$

The value of traffic handled $G \cdot A \cdot Q$

Loss of time for repeated calls $\eta \cdot \chi \cdot G \cdot A(1 - Q)$ χ = a constant

Cost of switches bn

The net utility $G \cdot A[Q - \eta\chi(1 - Q)] - bn$

will be maximum when

$$A_0\,[E_n - E_{n+1}] \simeq \frac{b}{G} \cdot \frac{1}{(1 - H)(1 + q + \chi\eta)}$$

Experience indicates that $q > 0$ and that therefore the increase in traffic and the reduction of repeated calls with improved accessibility requires an increase in the number of switches.

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On a reduction of A, equilibrium is attained in a similar manner.

An increase in the number of switches brings an increase in accessibility, which increases M and reduces S. For every number of switches, n, within reasonable limits, there is a state of equilibrium for which

$$A_n = M_n + S_n \tag{3.8}$$

and the number of switches should be chosen so that the value of the successful connections less the cost of switches less the cost of repeated calls is as great as possible.

To be able to arrive at this figure, one must know how the subscribers' need of traffic increases with accessibility and how the frequency of repeated calls diminishes. One must also know on what basis the congestion is to be calculated in the event that the traffic A contains repeated calls to a greater or lesser extent. In the absence of knowledge of these relations the problem cannot be attacked by exact means. But this constitutes no hindrance to the attainment of a satisfactory solution to the problem of evaluating the inconvenience to subscribers of failure to obtain connection.

In all calculations intended as aids in arriving at a decision, one should aim at as simple and clear rules as possible.

It will have been apparent from the foregoing that the initiated traffic is dependent on repeated calls and on the subscribers' traffic requirement, which in turn are dependent on accessibility and traffic losses. The volume of traffic thereby changes with the number of switches, which implies an undesirable complication in determining the switch requirement.

For this reason the traffic is defined by a frequency function $f(A)$ which does not change character with a change in the number of switches. This definition implies that the reduction of repeated calls with rising number of switches corresponds precisely to the increase in the subscribers' traffic requirement. In reality the reverse often holds, that the initiated traffic increases with improved quality of service, a fact which should be remembered when estimating the future frequency function $f(A)$ for the traffic.

After this first simplification of the problem it remains to decide how to calculate the subscribers' loss of time due to failure to obtain connection.

In a pure delay system this loss of time is equal to the total waiting time in the system. With full availability, and provided that the frequency function for the traffic can be represented by a single value of the traffic during one hour per day, the total mean waiting time is determined from the expression

$$A \cdot E_{2,\,n} \cdot \frac{1}{n - A}$$

where $E_{2,\,n}$ is Erlang's formula for delay systems.

In a pure lost call system the mean waiting time under the same assumptions is determined from the expression

$$y \cdot E_{1,\,n} \cdot \tau$$

where $E_{1,\,n}$ is Erlang's formula for lost call systems

τ = the mean loss of time caused by failure of connection

y = the call intensity

But, as already mentioned, whether designed as a delay or a lost call system, a plant functions in reality as a mixed system which cannot be exactly described until a study has been made of the subscribers' habits and reactions.

On this account, and since the traffic must be described by its frequency function, which in the case of delay systems leads to considerable mathematical difficulties, the author suggests the use of time congestion in lost call systems for assessing the inconvenience to subscribers. It is also assumed that, on encountering congestion, the subscriber loses the entire time which an average conversation takes.

Under these assumptions, *viz.*

1. that the initiated traffic is described by a frequency function independent of the accessibility
2. that the time congestion in lost call systems is taken as the basis for evaluation of inconvenience to subscribers
3. that on encountering congestion the subscribers lose the entire time taken by an average conversation

the number of switches can be computed as follows.

For an initiated traffic A, a time congestion $E(n, A)$ is obtained the magnitude of which is determined by the traffic A and the number of switches n.

$E(n, A)$ represents the part of the total time during which an individual subscriber cannot obtain connection, and the time lost by subscribers owing to congestion is represented by $y \cdot E(n, A) \cdot \tau$. According to assumption 3 above, $\tau =$ the mean conversation time s, and therefore the subscribers' loss of time is proportional to

$$A \cdot E(n, A)$$

Provided that the economic value of the inconvenience to subscribers caused by congestion and by the need for making repeated calls is in direct proportion hereto, the loss is determined from the following expression (cf. 2.1 in chap. 2):

$$G \cdot T \cdot \int_0^\infty A \cdot E(n, A) \cdot f(A) \cdot dA \qquad (3.9)$$

In this expression

G = quality of service factor as before (eq. 3.3), representing the capitalized value of one hour per day of the subscribers' time

T = the time expressed in hours per day described by the frequency function $f(A)$

$E(n, A)$ = the time congestion

The plant provision must be such that the sum of the capitalized cost of the plant plus the sum of the value of inconvenience to subscribers is as small as possible.

For a single switching group this means that the number of switches n must be chosen so as to reduce the following expression to a minimum:

$$K(n) = b \cdot n + G \cdot T \cdot \int_0^\infty A \cdot E(n, A) \cdot f(A) \cdot dA \tag{3.10}$$

Assume that Δn is the smallest number of switches which for technical reasons can be added to or subtracted from the switching group. Equation 3.10 will then be minimum for a value of n which satisfies the conditions

$$K(n \pm \Delta n) > K(n)$$

or written

$$T \cdot \int_0^\infty A \cdot [E(n - \Delta n, A) - E(n, A)] f(A) \cdot d(A) >$$

$$> \frac{b \cdot \Delta n}{G} \geq T \int_0^\infty A [E(n, A) - E(n + \Delta n, A)] f(A) \cdot d(A) \tag{3.11}$$

A numerical example of the use of this circuit provision formula is given in Annex 3.

For several switching groups the sum of the cost of switches and of the loss of time in the switching groups must be minimized in the same way, *i.e.* we must minimize the expression

$$K(n_1, \ldots . n_i, \ldots .) = \sum_j b_j \cdot n_j + \sum_i G_i \cdot T_i \cdot \int_0^\infty A_i \cdot E_i \cdot f(A_i) \cdot dA_i \tag{3.12}$$

An increase in the number of switches in a given switching group allows greater accessibility in that group and, consequently, there is greater congestion and loss of traffic in all subsequent switching groups. A reduction of the number of switches causes a corresponding lowering of traffic losses in subsequent switching groups. The optimum number of switches $n_1, n_2 \ldots$ is determined by a system of simultaneous difference equations, the number of which is equal to the number of switching groups. These equations are, of course, dependent on the particular trunking scheme; a general presentation of the equations will therefore be unduly complicated and they will not be shown here. In principle these equations have the same structure as eq. 3.11.

The numerical treatment of problems requiring minimization of expressions of the type of eq. 3.12 will be dealt with in Annex 4.

To the time costs dealt with above we should properly add a risk cost corresponding to the economic loss which may be incurred if a message cannot be transmitted at the proper time. For reasons similar to those given under 3.3.1, these risk costs are included in the constant G.

3.4 Economic optimum for conductor diameter and number of switches

General conditions

If the initiated traffic A, owing to the prolongation of conversation time, is increased to $A \cdot \vartheta_m$, the time cost to subscribers amounts to

$$K_{t_2} = \sum_i G_i \cdot T_i \cdot \int_0^\infty A_i \cdot \vartheta_m \cdot E_i(A_i \cdot \vartheta_m) \cdot f(A_i) \cdot dA_i \quad \text{kronor} \tag{3.13}$$

According to eq. 3.3 the value of inconvenience to subscribers under conditions of unsatisfactory transmission, provided that the risk cost is included in the service factor G, is

$$K_{t_1} = 2 \cdot G \cdot T_0 \cdot (\vartheta_m - 1) \cdot N \quad \text{kronor} \tag{3.14}$$

According to eq. 3.2 the part of the exchange cost which is dependent on the number of switches is

$$K_s = \sum_i b_i \cdot n_i \quad \text{kronor} \tag{3.15}$$

Finally, according to eq. 3.1, the cost dependent on the conductor diameter in the network is

$$K_a = c \cdot l_m \cdot z^2 \cdot N \quad \text{kronor} \tag{3.16}$$

These four categories of cost represent the part of the cost of the telephone plant that is dependent on conductor diameter and number of switches. The interplay between these costs and the cost determining factors is illustrated graphically in *Fig. 3.3*.

A necessary condition for the sum of the costs being a minimum is clearly

$$\begin{aligned} &\frac{\partial}{\partial z}(K_a + K_{t_1} + K_{t_2}) = 0 \\ &\frac{\partial}{\partial n_i}(K_s + K_{t_2}) = 0 \end{aligned} \tag{3.17}$$

In the latter equation $\frac{\partial}{\partial n_i}$ indicates that n_i must be so chosen that the difference of cost between $n_i - \Delta n_i$ and n_i switches is positive, whereas the difference of cost between n and $n_i + \Delta n_i$ switches is negative.

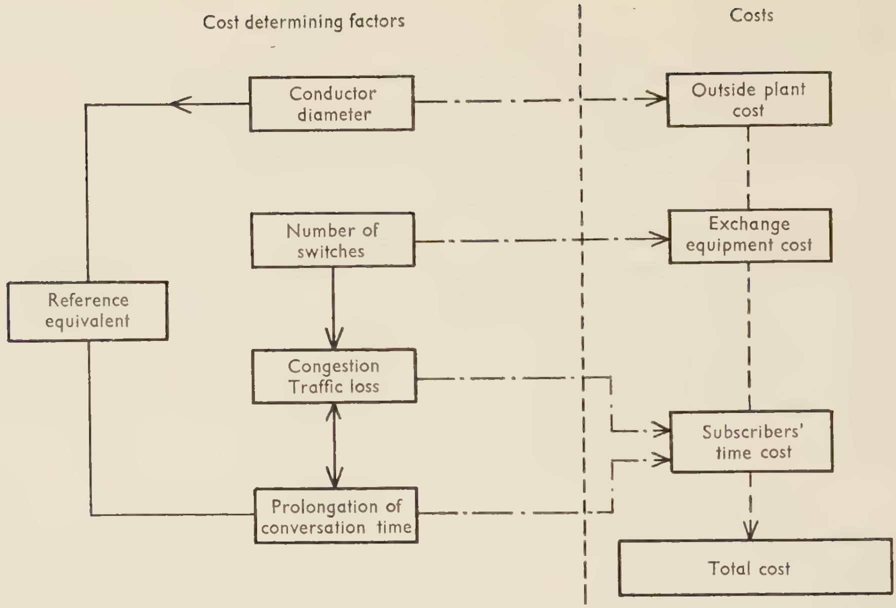

Fig. 3.3. Graphic representation of costs and cost determining factors in conjunction with the following problem:
Determine the conductor diameter and reference equivalent in a telephone cable plant, and the number of switches and the traffic losses in the exchange, for an isolated plant with given number of subscribers and traffic, so that the aggregate cost of outside plant, exchange equipment and loss of time to subscribers is as small as possible.

From this system of equations, as also from *Fig. 3.3*, it is apparent that the conductor diameter, and the reference equivalent and prolongation of conversation time dependent thereon, and the speech circuit and the number of switches and traffic losses dependent thereon, are mutually dependent and, in principle therefore, must be determined simultaneously. This adds a considerable complication, especially because the reference equivalent has for obvious reasons to be determined over a much longer period of time than the number of switches, which makes it of interest to investigate the degree of interdependence between the prolongation factor and the traffic losses. Fortunately the interdependence is slight, as shown in Annex 1.

CHAPTER 4

Different Methods of Estimating the Quality of Service Factor G Representing the Value of Subscribers' Time

In assessing the inconvenience to subscribers in respect of failed connections and prolongation of conversation time, we have so far made use of a factor G which may be said to represent the economic value of subscribers' time. In actual fact this constant is an irreducible social cost which quite naturally is somewhat difficult to estimate. G has a certain relation to other service factors for other public utilities such as bridges, roads or electricity supply. But this is of no help at present in establishing or estimating G.

For this reason we shall describe some of the methods which might be used for estimating this constant for existing requirements. It must be emphasized from the start, however, that the final establishment of this factor is a question of commercial policy. We shall deal with five different hypotheses, *viz.*

1. Time evaluation
2. Maximization of profits
3. Substitution
4. First cost
5. As-if principle

which will finally be summed up in a simple recommendation.

The presentation will be initially based on the conditions within a city network or small zone. Not until section 4.6 shall we expand the conditions to embrace long distance traffic.

4.1 Time evaluation

As starting point for this hypothesis we assume that a subscriber's time has a certain value, g kr/hour, which corresponds at least to his average hourly earnings. This subscriber's income is also assumed to derive from a given use of the telephone, *i.e.* we assume that the traffic initiated by each subscriber is known. We assume, furthermore, that the subscribers' average hourly earnings are constant during the relevant period and so obtain for G the expression

$$G = \frac{365 \cdot g}{r} \quad \text{kronor} \tag{4.1}$$

where

g = hourly earnings, kr/hour

r = interest factor

For $g = 8$ and $r = 0.08$, therefore,

$$G_1 = \frac{365 \cdot 8}{0.08} = 36{,}500 \text{ kronor}$$

If a subscriber converses for $T_0 = 0.4$ hour per day, the capitalized value of his time during the period in which he is using the telephone is $G_1 T_0 = 14{,}600$ kronor.

4.2 Maximization of profits

A telephone enterprise has certain tariffs representing, on an average, u kr/hour and wishes to dimension its plant so as to obtain the greatest possible profit. The value to be assumed by the constant G on the basis of such an aim is determined on the same principle as under 4.1. In this case we get

$$G_2 = \frac{365 \cdot u}{r} \quad \text{kronor} \tag{4.2}$$

where

u = call fee, kr/hour and subscriber

r = interest factor

For $u = 4$ and $r = 0.08$, we get for example

$$G_2 = \frac{365 \cdot 4}{0.08} = 18{,}250 \text{ kronor}$$

Since the value of subscribers' time should be greater than its cost, clearly

$$G_2 < G_1$$

from which it follows that a plant must have a greater number of switches if plant provision is to be based not only on obtaining the greatest possible profit for the telephone enterprise, but if it is also taken into account that the *value* of the use of the telephone should properly exceed its *cost*.

4.3 Substitution

We shall attempt to estimate the costs which would be incurred if the subscribers were to seek by other means to meet the requirement denied by the switching equipment. It need hardly be pointed out that this assumption is purely hypothetical. In reality, when a subscriber encounters a hindrance, he makes a new call in the great majority of cases. The assumption must therefore be considered solely as a step in the attempt to estimate G.

If the congestion on a circuit is high, it is conceivable that the subscribers would install special lines or even additional telephone exchanges to guarantee quick connection with important subscribers. In this case the situation might be compared roughly to a power transmission line which, under conditions of peak load, can be economically reinforced by a local power station.

Under conditions of low congestion, on the other hand, it does not pay subscribers to install private telephone lines or exchanges. The denied requirement could be substituted by sending a messenger or by other action which requires no investment but the cost of which is directly proportional to the number of actions required. In this case the factor G is determined by the expression

$$G_3 = \frac{365 \cdot h}{r \cdot s} \tag{4.3}$$

where

h = the cost per action for a hypothetical substitution of a denied call, kr
r = interest factor
s = mean time of conversation, hours

In order that this factor shall be equal to G_1, h must be equal to $s \cdot g$.

For $g = 8$ and $s = 0.05$, $s \cdot g = 0.40$ kr, which roughly corresponds to the postage of a letter.

It is thus clear that

$$G_3 > G_1 > G_2$$

A cost of 1 kr/action on encountering congestion means that, at $s = 0.05$,

$$G_3 = \frac{365 \cdot 1}{0.08 \cdot 0.05} = 91{,}250 \text{ kronor}$$

or more than twice as high as in the evaluation of subscribers' time at 8 kr per hour.

4.4 Plant cost

The constant G can also be estimated on the basis of plant cost if we assume the following principle:

In order that a telephone circuit shall be economically justified, the value of the traffic obtained on it must exceed the plant cost.

The cost of a telephone circuit can be divided into a fixed portion, a, irrespective of the quantity of traffic, and of a portion, $b \cdot n$, which is directly proportional to the number of switches, n, so that the plant cost may be written

$$a + b \cdot n$$

The cost per erlang hour will then be

$$S_0 = \frac{a + b \cdot n}{A_m (1 - E_m)} \tag{4.4.1}$$

where

$$A_m = 24 \int_0^\infty A \cdot f(A) \cdot dA \tag{4.4.2}$$

and

$$E_m = \frac{\int_0^\infty A \cdot E \cdot f(A) \cdot dA}{\int_0^\infty A \cdot f(A) \cdot dA} \tag{4.4.3}$$

i.e. the traffic losses in relation to the mean quantity of traffic.

In a telephone plant the fixed costs, here represented by the constant a, are very considerable compared with the variable costs, here represented by the expression $b \cdot n$. In addition the mean congestion, E_m, is generally very small.

Therefore as a preliminary good approximation we may put

$$S_0 \cong \frac{a + bn_0}{A_m} \tag{4.4.4}$$

where n_0 is a number of switches assumed to be in the neighbourhood of the optimum number.

According to the hypothesis the value of the subscribers' time should exceed its cost by, say, a factor $q > 1$. The number of switches can therefore be computed by minimizing the expression

$$b \cdot n + q \cdot S_0 \cdot A_m \cdot E_m \tag{4.4.5}$$

If the resulting value of n deviates greatly from the assumed value of n_0, the calculations must be repeated with a new value of n_0. It can be shown that this procedure converges very rapidly. Thus in this case we must use for G the value

$$G_4 = q \cdot S_0 \tag{4.4.6}$$

which can thus be compared with the values of G_1, G_2 and G_3 calculated in accordance with the preceding hypotheses.

If, for example, the cost of the telephone plant, including the present value of plant replacement and the cost of administration, operation and maintenance, amounts to 3,000 kronor per subscriber, and the total time of conversation is, on an average, 0.4 hour per day, G must be greater than

$$\frac{3{,}000}{0.4} = 7{,}500 \text{ kronor}$$

4.5 As-if principle and reconciliation of the hypotheses

The various hypotheses presented lead to fairly divergent values of the constant G, and *a priori* it is impossible to decide which of them is preferable or, alternatively, on what basis they should be compared. Therefore the establishment of this constant, which represents an irreducible social cost, involves certain difficulties which, however, are greatly alleviated since the reference equivalent and the number of switches are dependent only on the magnitude of the constant (see Annex 2).

Another way of estimating G is to examine a number of practical cases and from them to calculate the factor G which would provide precisely the same result in respect of, for example, switches and congestion as in the case under consideration. A calculation of this kind will give for each individual plant the answer that it has been constructed *as if* it were calculated for a given value of subscribers' time.

If, for example, a traffic flow of $A_0 = 30$ is handled by a group of 40 circuits and with a congestion of $E = 0.0144$ under conditions of full availability, this provision of plant at a cost of, say, 1,000 kronor per switch has been made *as if* the plant were dimensioned for a value of G obtained from the equation

$$\varepsilon = \frac{1{,}000}{G} = 30\,(E_{40} - E_{41}) = 0.003976 \cdot 30 = 0.1193$$

i.e. $G = 8{,}380$ kronor.

If one continues to examine a number of practical cases in a similar manner, one sometimes obtains values of G which vary in magnitude as 10:1, perhaps more, which raises the question whether any reason exists which could justify such great differences.

If, as has hitherto been done in practice, the frequency function $f(A)$ is represented by a single value A_0, certain threshold values for the congestion must be reached before a plant extension is undertaken. These values are small if the required investment relates to individual switches, and fairly large if, for example, a trunk cable is concerned. In the former case the threshold values may be of the order of one or less per mille, in the latter of one or less per cent, before a new extension is needed to bring down the congestion to a per mille level for A_0. In this case one may rightly ask whether it is advisable, even for a fairly short time, to allow so high a congestion and such large variations in the level of service. The administration may perhaps follow the principle of not allowing higher congestion than, say, 1 per cent for A_0 before an extension of a trunk route is undertaken. This standard has its counterpart in a fairly high value of G, indicating that the route has been extended *as if* that value had been used for planning of the extension in question.

Apart from these two different ways of establishing the magnitude of G, *viz.* of calculating it on the basis of practical cases both as regards the addition of individual switches and as regards the addition of entire plant units, the value of G which would give a particular reference equivalent can be calculated from the value of the reference equivalent used in practice.

Arguments of this kind, in conjunction with the hypotheses presented under 4.1—4.4, help in arriving at a probable *a priori* value of this factor which is of guidance in deciding commercial policy.

If we accept a value of G obtained in this way and dimension the plant accordingly, the plant will have a capacity corresponding to one and the same level of service. This applies not only to the provision, for example, of switches, but also in principle to the establishment of the quantity and date of plant extension and to the magnitude of the reference equivalent for a telephone circuit. We may thus say, that, at a given level of G, the congestion in a telephone exchange and the reference equivalent for a telephone network shall have specific values, and may thus refer such widely differing magnitudes as these to a common denominator G.

4.6 Quality of service factor G in long distance traffic

A fundamental assumption for the determination of G is, as already stated, that the value of a telephone circuit shall always exceed its cost.

Since the cost of a long circuit is greater than that of a short, the principle implies that the value of the subscriber's time, represented by the service factor, must be greater for a long than for a short circuit provided that G in both cases is obtained by multiplying the first costs by the same factor q.

As against this conclusion one may object that the value of a telephone connection is by no means dependent on its length, since conversation over short distances can have at least as great a value as over long distances, and that therefore the service factor must be a cost independent of distance. If one holds this opinion and does not wish to depart from the principle that the value of a circuit shall exceed its cost, one must conclude that the service factor G must be decided in relation to the cost of the longest and most expensive national circuit and to its value. This means that the magnitude of G must be determined from the marginal cost of extending a national telephone circuit and that therefore the same value of G must be used for switch provision both on local and long distance circuits.

As against this view, however, there is the serious objection that long distance and local traffic are two essentially different services offered to subscribers of entirely different categories.

The author has therefore come to the conclusion that switches and circuits for long distance traffic must be provided on the basis of a value of the service factor G which is in proportion to the marginal cost of a long distance circuit, whereas switches and circuits for local traffic should be provided on the basis of a lower value of G which bears a certain relation to the cost of local traffic.

The final establishment of these relations, *i.e.* of G, is a question of commercial policy, the decision in regard to which should be assisted by a discussion of the kind outlined in this chapter.

CHAPTER 5

The Inconveniences Attendant on Use of the Telephone

A telephone conversation cannot be started at the moment when a subscriber decides to call another subscriber, since it takes some time to build up the connection. This preparation involves a loss of time and therefore a certain inconvenience to subscribers.

These inconveniences may be divided into two categories:

A. Inconveniences which may increase or decrease without altering the technical structure of the telephone system.

B. Inconveniences which cannot be changed unless the system is changed or added to.

Among inconveniences of the first kind are congestion, intelligibility and fault rate, which within any one system can be changed by changing the number of switches in the exchanges, the diameters of conductors in the cable plant or by changed maintenance effort.

Among inconveniences of the latter type are the time required by the caller to reach his correspondent on condition that no congestion or technical fault exists. This inconvenience is dependent on the time taken to ring up the other party, on how often the called party is engaged or fails to answer, and finally on the extent to which the caller dials the wrong number.

These inconveniences can only be reduced by altering or adding to the system, by reducing the switching time, by installing special equipment at subscribers' premises showing the origin of calls, or finally by using a pulsing system which causes less wrong connections than, for example, the dial.

It may be of interest to illustrate these inconveniences numerically on the basis of an evaluation of the loss of time to subscribers.

A 1. Congestion

Say that subscribers lose, on an average, 2 seconds per day owing to traffic losses in switches and that the value of subscribers' time is 8 kr/hour. Capitalized at 8 % this loss of time amounts to

$$\frac{365 \cdot 8 \cdot 2}{0.08 \cdot 3{,}600} \text{ or approx. } 20 \text{ kr/subscriber}$$

This amount invested in a 500-point selector group of 10,000 lines corresponds to about 200 line finders + 200 final selectors + 20 registers.[1]

[1] The investments and quantities of equipment stated under A1—3, corresponding to time losses, must not, of course, be taken as meaning that the measures are or are not economically justified, but are intended solely to give a picture of the importance of the seconds lost in terms of money and quantities of equipment.

A 2. Prolongation of conversation time

Say that on a trunk call a subscriber loses 0.5 second per day owing to unsatisfactory transmission and that the value of the subscriber's time under conditions of trunk traffic is 80 kr/hour. Capitalized at 8 % this corresponds to

$$\frac{365 \cdot 80 \cdot 0.5}{0.08 \cdot 3{,}600} \text{ or about } 50 \text{ kr/subscriber}$$

This amount invested in cable plant averaging 1 km length of line would imply that the conductor diameter can be increased from 0.4 to 0.5 mm.

A 3. Technical faults

Say that 0.25 % of local calls fail owing to technical faults on the circuits. With 5 calls per day and a time loss of 200 seconds per fault, this represents a time loss of $0.0025 \cdot 5 \cdot 200 =$ 2.5 seconds per day, or capitalized at 8 % and 8 kr/hour,

$$\frac{365 \cdot 8 \cdot 2.5}{0.08 \cdot 3{,}600} = 25 \text{ kr/subscriber}$$

Say, too, that 1 % of *trunk calls* fail owing to technical faults. With 0.25 call per day and a time loss of 200 seconds per fault this corresponds to $0.01 \cdot 0.25 \cdot 200 = 0.5$ second per day, or capitalized at 8 % and 80 kr/hour

$$\frac{365 \cdot 80 \cdot 0.5}{3{,}600 \cdot 0.08} = 50 \text{ kr/subscriber}$$

These amounts invested in plant for 100,000 subscribers mean that the control equipment, for example, could be increased by 4 mill. kr. and the maintenance staff by 5 men earning 8 kr/hour.

The losses of time indicated under A 1—3 may be changed by a change in the number of switches, in the conductor diameter or in the equipment and staffing of the maintenance service.

B. Time of setting up a call

The losses described in the following paragraphs, associated with the time taken to set up a call, cannot be changed without the system being altered or the subscribers altering their behaviour pattern.

B 1. Called subscriber answers

The subscriber suffers a certain loss of time represented by the time taken to reach the telephone, look up the number in the directory or ascertain it from Directory Enquiries, raise the handset, wait for dial tone, dial the number, wait for the ringing signal and for the called subscriber to answer.

Say that 70 % of calls are put through immediately and that the called party answers 30 seconds after the caller has decided to ring him. With 5 calls per day this corresponds to 150 seconds per day, or capitalized

$$0.70 \cdot \frac{365 \cdot 8 \cdot 150}{0.08 \cdot 3{,}600} \text{ or about } 1{,}000 \text{ kr/subscriber}$$

At most half of this loss of time derives from the operating time of the system, *i.e.* the time taken to raise the handset, wait for dial tone, dial the number and wait for the ringing signal, say 10—15 seconds per call. Having regard to the value of a subscriber's time, any shortening of this operating time is of interest even if it must be purchased at the cost of increased investment.

Every second by which the operating time can be reduced implies a gain of 5 seconds per day at 5 calls per day or, capitalized on the same basis as previously, 50 kr/subscriber. If 8 kr/hour is accepted as the value of subscribers' time, it is worth while spending an amount in the order of 50 kr/subscriber for every second by which the operating time can be reduced.[1]

B 2. Called subscriber does not answer

The aforementioned losses of time are greater if the called subscriber does not answer, since the caller usually waits during several ringing signals to make sure that the person is not at home.

Say that the called subscriber fails to answer on 12 % of calls and that altogether 50 seconds elapse before the caller replaces his handset. This corresponds to $50 \cdot 5 = 250$ seconds, or capitalized,

$$0.12 \cdot \frac{365 \cdot 8 \cdot 250}{0.08 \cdot 3{,}600} \text{ or about } 300 \text{ kr/subscriber}$$

The fact that the called subscriber does not answer usually results in the caller making one or more repeated calls.

If a piece of equipment (a telephone answerer) is installed at the subscriber's premises to issue a message as to when he will be available, so saving the caller from, for example, one repeated call, the equipment may be considered to be worth 300 kr/subscriber if a value of 8 kr/hour is put on the subscriber's time.[2]

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[1] A shortening of the operating time implies a reduction of the initiated traffic. This circumstance, added to the increased cost of switches in a more rapid system, means that the number of switches should be reduced as compared with a slower system. This must be taken into account, as also must the traffic losses, in an exact comparison between different systems. But this would carry us too far from our present context.

[2] Equipment of this kind can naturally be of value also for the called subscriber.

B3. Called subscriber engaged

If the called subscriber's telephone is engaged, the caller immediately hears busy tone. The loss of time will therefore be rather less than under B1, say 20 seconds. Assuming that the called subscriber is engaged on 10 % of the calls made to him, this represents a capitalized cost of

$$0.10 \cdot \frac{365 \cdot 8 \cdot 20 \cdot 5}{0.08 \cdot 3{,}600} \quad \text{or about 100 kr/subscriber}$$

A device which informs the caller as soon as the called party's telephone is free is therefore worth considerably less to subscribers than a device which announces when the called party will be available for conversation.

B4. Wrong connection

On a wrong connection the subscriber loses at least as much time as when a correctly connected call is answered. If this happens on 5 % of calls, the wrong connection corresponds to a capitalized value of

$$0.05 \cdot \frac{365 \cdot 8 \cdot 150}{0.08 \cdot 3{,}600} \quad \text{or about 75 kr/subscriber}$$

A pulser which reduces the risks of dialling a wrong number by one third is therefore worth an extra cost of 25 kr/subscriber. (Concerning the effect of a more rapid pulser, see B1 above.)

The aforementioned estimates are approximate. But they are sufficiently accurate to show that the inconveniences due to congestion, poor quality of transmission or poor maintenance should be less than one-tenth of all inconveniences attaching to use of the telephone. The types of inconvenience referred to under A and B, however, are not commensurable and therefore allow no conclusion to be drawn as to whether, for example, congestion may be allowed to increase since it is in any case submerged in other inconveniences.

If, for example, the called subscriber fails to reply or is engaged, the caller receives information, *viz.* that the called party is not available and that he must wait to obtain connection. In the event of congestion or a technical fault the caller is given no information whatsoever concerning the called party. These two kinds of inconvenience can, therefore, not be compared quantitatively.

Nor can the part of the calling time which is independent of the operating time of the system be directly compared with any of the remaining inconveniences, since the caller is already aware of this fact and his resistance to this form of inconvenience has been overcome as soon as he decides to use the telephone. Nor is it specific of telephony that it takes time to obtain connection with another person.

Again, insofar as they are independent of congestion, operating times cannot be compared with other inconveniences. They are of some interest, however, in comparing different systems or for assessing the profitability of a proposed measure which reduces the operating time, such as the use of quicker switches, a telephone answerer or keysets, and perhaps as guidance for determining to what extent these and other facilities shall be allocated to different classes of employees. In the latter case the value of G will, of course, have to be put higher for the top management.

Our argument has been based on the fact that the telephone system is made up of certain predetermined standard components, and we have shown in principle how the number of switches and the reference equivalent of circuits can be determined on the basis of the value of subscribers' time.

But the technically and economically important question of technical faults and their control has had to be left aside as being a separate matter requiring special treatment. The same applies to the question of operating times, which are associated with questions of system and can only be dealt with in connection with design models of which the principles at least have been worked out.

In a telephone plant consisting of specific components and designed to a specific system, it is important to devote special attention to congestion, intelligibility and fault rate, since any inadequacy in any of these three measures of the quality of a telephone connection implies serious inconvenience to subscribers.

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Simplification of the Problem of Determining Cable Conductor Diameter and Number of Switches by Considering It as Two Separate Problems

The determination of reference equivalent and the number of switches in a telephone plant is, as shown in chapter 3, a simultaneous problem which would be fairly complicated if congestion and prolongation of conversation time were closely interdependent.

In actual fact they are little so, as will be shown in the sequel.

It was found in chapter 3 that the sum of the costs of outside plant and exchange equipment required for the calculations, and the value of the inconvenience to subscribers due to prolongation of conversation time and congestion, can be written

$$K(z, n) = [c \cdot l_m \cdot z^2 + 2 \cdot G \cdot T_0(\vartheta_m - 1)] \cdot N + \sum_j b_j \cdot n_j + \sum_i G_i \cdot T_i \int_0^\infty A_i \cdot \vartheta_m \cdot E_i \cdot f(A_i) \cdot dA \tag{1}$$

In this annex, which is intended solely to show that the determination of reference equivalent and congestion can be treated as two separate problems, we may assume that the switching groups in the exchange are replaced by a single group with full availability. Studies have shown that this can be done with sufficient accuracy as regards the results attainable in this context.

On the assumption that the frequency function $f(A)$ describes the traffic during one year the above expression can therefore be written

$$K(z, n) = [c \cdot l_m \cdot z^2 + 2\,G \cdot T_0(\vartheta_m - 1)] \cdot N + b \cdot n + G \cdot 24 \int_0^\infty A \cdot \vartheta_m \cdot E \cdot f(A) \cdot dA \tag{2}$$

where

$$E = E_{1n}(A \cdot \vartheta_m)$$

according to Erlang's formula for time congestion, E_{1n}.

Derivation in respect of z gives

$$\frac{\partial K}{\partial z} = \left[2 \cdot c \cdot l_m \cdot z + 2\,G \cdot T_0 \cdot \frac{\partial \vartheta_m}{\partial z}\right] N + G \cdot 24 \int_0^\infty \frac{\partial (A \cdot \vartheta_m \cdot E)}{\partial z} \cdot f(A) \cdot dA \tag{3}$$

Since the congestion can be calculated from Erlang's formula,

$$\frac{\partial (A \cdot \vartheta_m \cdot E)}{\partial z} = \frac{\partial \vartheta_m}{\partial z} \cdot A \cdot E[1 + n - A \cdot \vartheta_m (1 - E)] \tag{4}$$

the minimum condition can be written

$$\xi = \frac{c \cdot l_m}{(1+\delta) G \cdot T_0} = -\frac{1}{z} \cdot \frac{\partial \vartheta_m}{\partial z} \tag{5}$$

where

$$\delta = \frac{24}{2 T_0 \cdot N} \int_0^\infty A \cdot E \cdot [1 + n - A \cdot \vartheta_m (1 - E)] \cdot f(A) \cdot dA \tag{6}$$

Since the total initiated traffic $N \cdot T_0$ is

$$N \cdot T_0 = 24 \int_0^\infty A \cdot f(A) \cdot dA \tag{7}$$

this expression can be written

$$\delta = \frac{\int_0^\infty A \cdot E \cdot [1 + n - A \cdot \vartheta_m (1 - E)] \cdot f(A) \cdot dA}{2 \int_0^\infty A \cdot f(A) \cdot dA} \tag{8}$$

The number of switches is usually so large that a traffic $A = 2n$ occurs very seldom. $E(1 + n - A(1 - E))$ is in this case of an order of magnitude of 0.5.

The traffic which is subjected to congestion is roughly 10 % of the total traffic.

An upper limit for the factor δ according to eq. 8 is therefore

$$\delta < \tfrac{1}{2} \cdot 0.5 \cdot 0.10 = 0.025$$

If in eq. 5 we alter the factor ξ accordingly, *i.e.* by about 2.5 %, we shall find that there are very negligible changes in conductor diameter and reference equivalent, and that these magnitudes can be determined practically independently of the traffic losses.

To determine the number of switches, we form by means of eq. 2 the cost difference between $n - 1$ and n and between n and $n + 1$ switches, which gives the minimum condition

$$24 \int_0^\infty A \cdot \vartheta_m (E_{n-1} - E_n) \cdot f(A) \cdot dA > \frac{b}{G} \geq 24 \int_0^\infty A \cdot \vartheta_m (E_n - E_{n+1}) \cdot f(A) \cdot dA \tag{9}$$

where, as before,

$$E_n = E_{1n}(A \cdot \vartheta_m)$$

The prolongation factor ϑ_m is of an order of magnitude of 1.01.

This is sufficient to increase the number of switches somewhat on routes carrying heavy traffic.

From the above it is apparent that the calculations can be carried out in the following steps.

Step 1. Conductor dimensions, prolongation factor and reference equivalent are calculated from the expression

$$\xi \simeq \frac{c \cdot l_m}{G \cdot T_0} = -\frac{1}{z} \cdot \frac{\partial \vartheta_m}{\partial z} \tag{10}$$

Step 2. The number of switches is calculated on the basis of the frequency function $f(A)$ of the estimated future traffic corrected by the value of ϑ_m obtained in step 1.

Step 3. The factor δ is calculated according to eq. 6 or 8, and with the assistance of eq. 5 a check is made that ϑ_m has suffered no such change as to necessitate an iteration as in steps 1 and 2.

ANNEX 2

Dependence of Result on Estimate of Quality of Service Factor, Traffic and Rate of Interest

It has been apparent from chapter 4 that the constant G, which is determinative of the transmission level and congestion, is an irreducible social cost the exact determination of which involves certain difficulties.

It is therefore of interest to ascertain the degree to which a change in this factor will affect the switch requirement and the transmission level.

1. The number of switches on a traffic route, on which the traffic is determined by the frequency function $f(A)$, and the time T hours per day described by $f(A)$ are, as we have seen from chapter 3, determined by the relationship

$$\varepsilon = \frac{b}{G} \tag{1}$$

which indicates the relationship between the cost per switch (b) and the value of the subscriber's time (G). The factor ε, which is dimensionless, may be regarded as a switch provision factor.

To obtain a preliminary survey, we assume that the frequency function $f(A)$ can be represented by a single value, A, of duration 1 hour and that use can be made of Erlang's formula for groups of circuits with full availability.

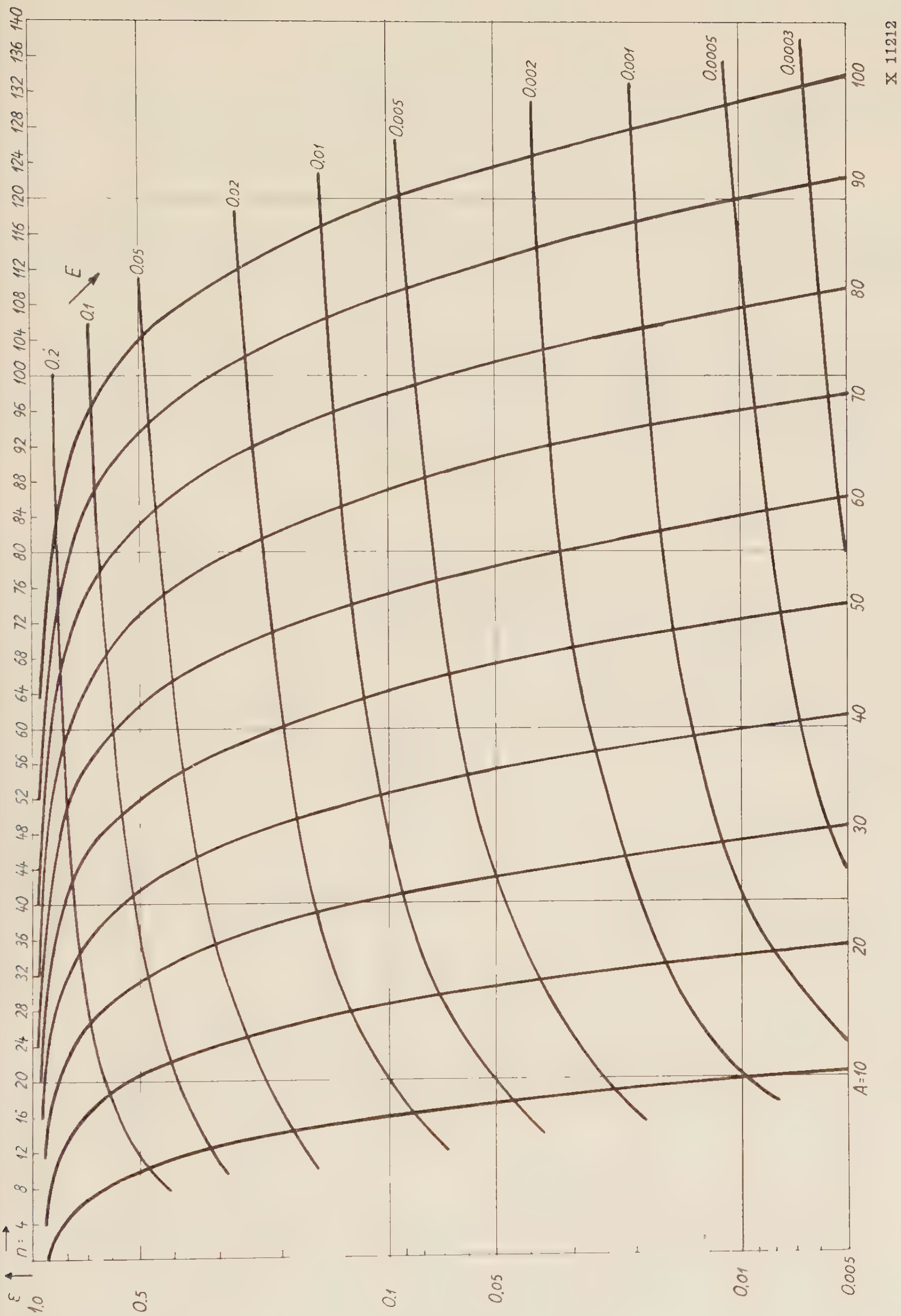

Fig. 1. The number of switches, n, and the congestion $E(n, A)$ as function of the switch provision factor $\varepsilon = \frac{b}{G}$.
$A = 10, 20, 30 \ldots 100$.

The number of switches can in this case be determined from tabulated values of Moe's improvement function[1]

$$F_{1,n} = A\,[E_{1,n}(A) - E_{1,n+1}(A)] \tag{2}$$

by finding the value of n which satisfies the inequality

$$F_{1,n-1} > \varepsilon \geq F_{1,n} \tag{3}$$

Fig. 1 shows the process of this function for $A = 10, 20, \ldots.\ 100$, and the congestion E under optimal conditions of switch provision for a given value of the switch provision factor ε.

It will be seen from this figure that the number of switches for a given traffic A must be increased as soon as ε decreases. For one and the same value of ε the congestion will be higher at a low than at a high traffic load. The congested traffic AE, however, grows with traffic A.

For $\varepsilon = 1$, *i.e.* when the switch cost b is equal to G, the number of switches is zero and the plant should not be constructed.[2] In this case, obviously, the subscribers can satisfy their communication requirements more economically by other means than through the telephone.

As soon as $\varepsilon < 1$, the number of switches initially grows very rapidly with diminishing ε. As ε diminishes still further, the growth becomes slower so that

$$\lim_{n \to \infty} \frac{n \cdot \log n}{-\log \varepsilon} = 1 \tag{4}$$

The following table shows how the number of switches grows with growing $\frac{G}{b}$ for $A = 10$ and $A = 50$.

G/b	$A = 10$		$A = 50$	
	n	E	n	E
1	0	1	0	1
10	16	2.2×10^{-2}	64	0.8×10^{-2}
10^2	20	1.9×10^{-3}	72	0.7×10^{-3}
10^3	23	1.8×10^{-4}	78	0.6×10^{-4}
10^4	26	1.1×10^{-5}	83	0.5×10^{-5}

[1] A Jensen: *Moe's principle*, Copenhagen 1950.

[2] At low traffic it becomes uneconomical to construct the plant as soon as $\varepsilon > \frac{A}{1 + A}$. Assume that a circuit carrying a traffic of 0.05 costs 1,000 kr/subscriber. G must then be greater than 20,000 kr if the plant is to pay its way.

Usually the switch provision factor is

$$0.01 < \varepsilon < 0.1$$

An analysis of the relationship illustrated in *Fig. 1* shows that the number of switches can be approximately calculated from the formula

$$n = C_A\left(A - \sqrt{A}\,{}^{10}\log \varepsilon\right) \tag{5}$$

where C_A can be considered a constant within a certain domain for the traffic $A_1 < A < A_2$.

This approximate formula is interesting in the respect that it indicates the formula for determination of the number of switches, which was set up by P V CHRISTENSEN[1] as early as 1913, *viz.*

$$n = A + \lambda\sqrt{A}$$

where λ is a value of a normal deviate.

The introduction of the factor ε, which is the relationship between the capitalized cost of a switch and the factor G, which represents the capitalized value of 1 hour per day of the subscriber's time, illustrates the implication of λ, which should have some interest apart from purely historical considerations. The formula shows that for an initiated traffic of, say, $A = 50$, the number of switches must be increased by about 2 if the factor G is doubled and that a switching stage which costs twice as much as a stage with the same initiated traffic must have about 2 switches less than the cheaper stage.

This old formula can thus serve as a useful rule of thumb for purposes of operations analysis and be employed at least for rough calculations in cases which fall within its sphere of validity.

Eq. 5 is not sufficiently accurate for exact determination of the number of switches even on the simplified assumption that the frequency function $f(A)$ can be replaced by a single value A representing one hour. The switch provision should not be established, however, on the basis of the traffic during any given hour but of the frequency function $f(A)$ for the traffic A. In this case the switch provision will depend on the form of this curve, especially for high values of A, and will therefore be decided by the tail of the frequency function. Studies have shown that, even under these assumptions, the number of switches can within certain limits be changed proportionally to log G. But in such case the proportionality factor will be greater.

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[1] P V CHRISTENSEN: *The Number of Selectors in Automatic Telephone Exchanges*, published in "Elektroteknikern" 1913, p. 207, also in Post Office Electrical Journal, 1914, p. 271.

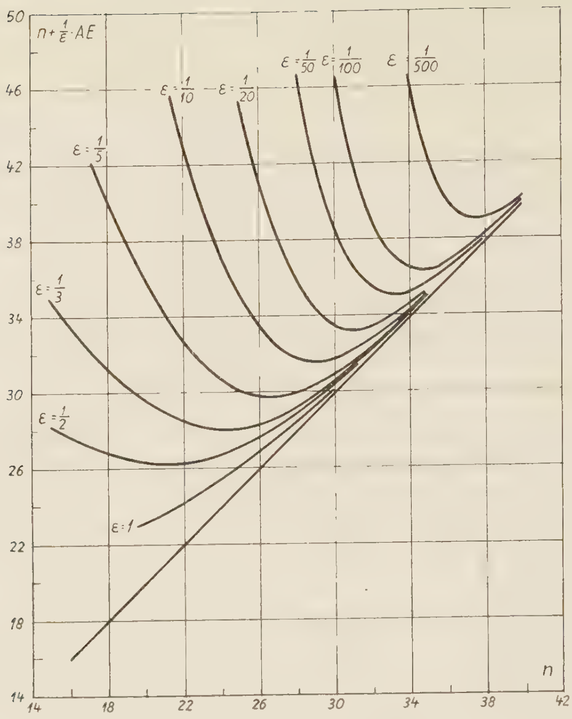

Fig. 2. The course of the function

$$f(n) = n + \frac{1}{\varepsilon} \cdot A \cdot E_{1,\,n}(A)$$

for $A = 20$ and different values of ε.

As in most problems of economic optimum, the curve of minimum cost is fairly flat. By way of example *Fig. 2* shows the function

$$n + \frac{1}{\varepsilon} \cdot A \cdot E \tag{6}$$

for $A = 20$ and different values of ε.

2. Apart from G, the result is of course also dependent on the accuracy with which the future traffic A can be estimated. From the approximate expression

$$\frac{d(AE)}{dA} \cong E(1 + n - A)$$

the dependence of the result on A is clearly fairly large.

Assume, for example, that the traffic has been estimated as $A = 30$ and the switch provision factor $\varepsilon = \frac{b}{G}$ as 0.045. The number of switches must then be $n = 44$ according

to Moe's improvement function. The table below gives the number of switches which should have been installed for $A = 29$, 30 and 31 and $\varepsilon = 0.03$, 0.045 and 0.060.

$\varepsilon \backslash A$	29	30	31
0.03	42	43	44
0.045	43	44	45
0.060	44	45	46

Thus an error of $\pm 3\%$ in the traffic estimate has the same influence on the result as an error in the estimate of G in the ratio 1 : 2.

3. A study of how the reference equivalent of a telephone circuit changes with the factor G requires more comprehensive calculations, which will be described in a future paper, *The Reference Equivalent within a National Network*. For a preliminary survey of this question we shall presume that the frequency function for the geographical distribution of subscribers is a constant, hence

$$f(x) = \frac{1}{L} \qquad (0 < x < L) \tag{7}$$

and that the prolongation factor can be written

$$\vartheta \simeq 1 + \gamma (R - R_*)^2 \tag{8}$$

Under these assumptions the mean prolongation factor

$$\vartheta_m = 1 + \frac{\gamma}{L^2} \iint_{\Omega} (R_{yx} - R_*)^2 \cdot dy \cdot dx \tag{9}$$

and the expression

$$-\frac{1}{z} \cdot \frac{\partial \vartheta_m}{\partial z} \tag{10}$$

will be explicitly determined by straightforward calculus. *Fig. 3* illustrates the result of some numerical computations made under the abovementioned simplified assumptions. From this figure it is evident that the overall reference equivalent

$$R = (2\alpha + \alpha_m) \cdot L + R_0 \tag{11}$$

is of the order of magnitude

$$R \simeq c_1 + c_2 \, {}^{10}\log \xi \tag{12}$$

with

$$\xi = \frac{c \cdot l_m}{G T_0}$$

(Cf. Annex 1, eq. (10)). The constant c_2 is about 6. This means that if G is doubled the reference equivalent will be decreased by about 2 db.

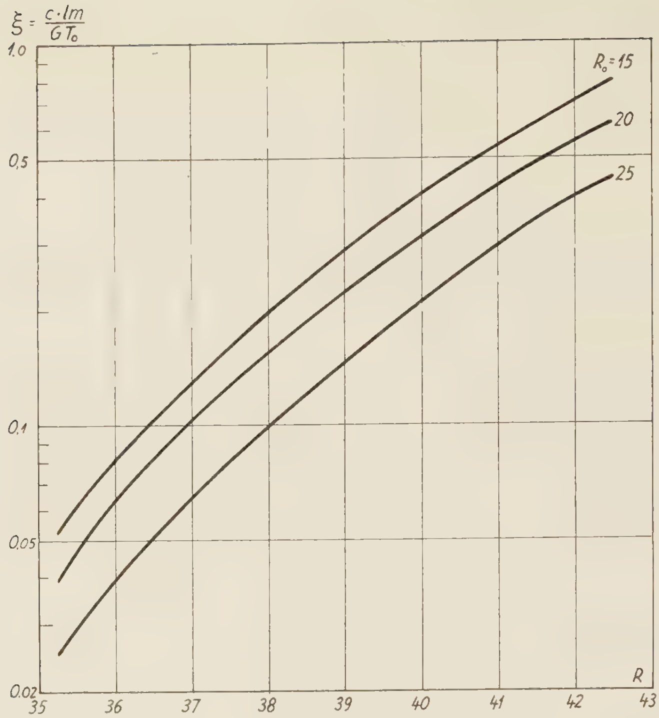

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Fig. 3. The overall reference equivalent R as function of

$$\xi = \frac{c \cdot l_m}{GT_0} = \frac{\textit{cost per mm}^2 \textit{ copper in subscribers' network}}{\textit{economic value of the conversation time}}$$

$R_0 = 15,\ 20,\ 25$ db, $L = 5$ km.

Thus both the number of switches and the reference equivalent are affected solely by the order of magnitude of the constant G.

4. The result is also affected by the rate of interest employed.

The switch cost b must include the present value of all plant costs, replacement of plant units and expenditure on operation and maintenance at the present value factor

$$\mu = 1 + \frac{1 - s}{(1 + r)^{t_a} - 1} + \frac{u}{r} \tag{13}$$

where

t_a = period of usage, years
r = interest factor
u = annual operations and maintenance cost in relation to plant cost
s = scrap value in relation to plant cost

The constant G includes the capitalization factor $\frac{1}{r}$. If the switch provision factor ε has a given value ε_0 at an interest factor r_0, then ε_0 changes to ε_1 at an interest factor r_1 according to the formula

$$\varepsilon_1 = \varepsilon_0 \cdot \frac{\mu_1 \cdot r_1}{\mu_0 \cdot r_0} \tag{14}$$

The table below shows the relationship $\frac{\varepsilon_1}{\varepsilon_0}$ for a period of usage of $t_a = 20$ years, $r_0 = 8\%$ and $r_1 = 4$, 8 and 15 %, $u = 0.02$, $s = 0$.

r_1	$\frac{\varepsilon_1}{\varepsilon_0}$
4	0.77
8	1.00
15	1.48

The table shows that the switch provision factor ε is doubled for a change in rate of interest from 4 % to 15 % and that the rate of interest can therefore be varied within fairly wide limits without appreciably changing the result.

The rate of interest used in this study is 8 %, which is a normal loan rate, 6 %, + risk interest 2 %.

The rate of interest cannot, however, be established once and for all, but the circumstances must be considered in each particular case.

If we imagine that the investments are financed out of loans, the calculated rate of interest should be at least equal to the interest on the last loaned capital.

When the corporation's own capital is used, the interest on the capital should be at least equal to the normal yield on telephone operations.

The amount by which the interest should be increased over and above the figure indicated will depend partly on the risk that is subjectively ascribed to the investment and partly on the supply of capital.

With a limited supply of capital the interest should be increased so much that the available capital is just absorbed by the economic degree of expansion of the plant components calculated at that rate of interest. In such cases a calculated rate of interest of 20 % or above may be warranted.

ANNEX 3

Calculation of the Number of Switches on a Traffic Route with a Single Switching Stage. Discussion on Basis of a Numerical Example

To determine the number of switches, n, on a traffic route, the following data are required, as indicated in chap. 3.

1. Cost b kr/switch.
2. Quality of service factor G kr, representing the capitalized value of 1 hour per day of the subscriber's time.
3. A frequency function $f(A)$ describing the traffic during an entire year.

The expression which is determinative of n, *viz.*

$$\varepsilon = \frac{b}{G} \simeq T \int_0^\infty A\,[E(n, A) - E(n+1, A)] \cdot f(A) \cdot dA \tag{1}$$

cannot be solved explicitly in terms of n. To determine the number of switches, therefore, it is necessary to assume various values of n on trial, to calculate the integral and to compare its value with $\varepsilon = \frac{b}{G}$.

For calculation of the integral the frequency function $f(A)$ is divided into classes of suitable extent so that certain probabilities $P(A_1)$, $P(A_2)$ .... correspond to the traffic values A_1, A_2 .... The right-hand term in eq. 1, the *improvement factor*, will thus be a sum

$$F = T \sum_A A \Delta E \cdot P(A) \tag{2}$$

where $A \Delta E$ is an abbreviation of the expression

$$A\,[E(n, A) - E(n+1, A)]$$

For numerical calculation of F by means of eq. 2, it is not necessary to include all traffic values in the classification for the entire year since low traffic values contribute negligible quantities to the improvement factor F.

As was shown in chap. 3, T is the time expressed in hours per day described by the frequency function $f(A)$. If $f(A)$ embraces the whole year, then obviously T is equal to 24. If the calculations include, for example, only such part of the tail as corresponds to the area 1/6, T will obviously be equal to 4 if the traffic during that period is described by an associated frequency function.

In our numerical example we make the following assumptions:

1. Cost per switch b = 1,000 kronor
2. Quality of service factor G = 25,000 kronor
3. A classification of the frequency curve for traffic values $A \geq 25$ and embracing 1/6 of the frequency curve for the entire year on the basis of the values in *Table 1.*

Table 1. Traffic distribution during $T = 4$ hours per day and $A \geq 25$. Traffic losses for $n = 49$.

A	$P(A)$	$A \cdot \Delta E$	$P(A) \cdot A\Delta E$
25	0.02815	0.000072	0.000002
26	0.09007	0.000234	0.000021
27	0.15312	0.000513	0.000079
28	0.18374	0.001140	0.000209
29	0.17455	0.002316	0.000404
30	0.13961	0.004421	0.000617
31	0.09774	0.007969	0.000779
32	0.06143	0.01360	0.000835
33	0.03532	0.02208	0.000780
34	0.01884	0.03416	0.000644
35	0.00942	0.05056	0.000476
36	0.00445	0.07178	0.000319
37	0.00200	0.09807	0.000196
38	0.00086	0.1293	0.000111
39	0.00036	0.1651	0.000059
40	0.00014	0.2047	0.000029
41	0.00006	0.2474	0.000015
42	0.00002	0.2914	0.000006
43	0.00001	0.3365	0.000003

Full availability conditions are assumed, so that Erlang's formula for lost call systems can be employed.

For determination of the number of switches it is advisable first to attempt to establish a preliminary lower limit. This can usually be obtained on the basis of the mean value of the traffic for the traffic distribution under consideration, in this case from the mean of the distribution for $25 \leq A < \infty$ in *Table 1.* This mean value is $A_m = 29$.

Provided that the contribution to F (eq. 2) for $A < 25$ is negligible and that the traffic is represented by a single value $A_m = 29$ during a period $T = 24/6 = 4$ hours per day, we obtain for $n = 45$, $n = 46$ and $n = 47$:

$$F_{45} = 4 \cdot 29\,[E(45, 29) - E(46, 29)] = 0.059$$

$$F_{46} = 4 \cdot 29\,[E(46, 29) - E(47, 29)] = 0.038$$

$$F_{47} = 4 \cdot 29\,[E(47, 29) - E(48, 29)] = 0.024$$

On the assumptions in the example $\varepsilon = \frac{b}{G} = \frac{1{,}000}{25{,}000} = 0.04$, and the preliminary calculation thus shows that the number of switches should probably be greater than 46.

We then make an exact calculation of the expression

$$F = 4 \sum_{25}^{\infty} A \Delta E \cdot P(A)$$

for $n = 47$, 48 and 49 switches. The calculation for $n = 49$ is shown in *Table 1*. The result is

$$F_{49} = 4 \cdot 0.00558 = 0.022$$
$$F_{48} = 0.033$$
$$F_{47} = 0.048$$

Hence the most economic number of switches is $n = 48$.

If the probable distribution of traffic, and not only its mean value $A_m = 29$, is taken into account in calculating the number of switches, the number increases by 2 from 46 to 48.

Thus if we wish to represent the traffic distribution by a single value applying to a given time per day, we cannot use the mean value but must make a certain addition to it.

In the present case an "equivalent" traffic of, for example,

$A_{*4} = 30$ applying during 4 hours per day

or

$A_{*1} = 33$ applying during 1 hour per day

gives the same number of switches as calculated from the traffic distribution, *i.e.* $n = 48$.

The traffic distribution dealt with in the example, the form of which accords closely with the tail of that illustrated in *Fig. 2.1*, chapter 2, can be mathematically described by the negative binomial distribution

$$P(k, r, p) = \binom{r + k - 1}{k} \cdot p^r \cdot q^k \qquad (k = 0, 1, 2 \ldots) \quad (p + q = 1) \quad (3)$$

In the example under consideration

$$p = 0.8, \quad q = 0.2, \quad r = 16$$

which implies that the mean value is

$$\alpha_1 = \frac{r \cdot q}{p} = 4$$

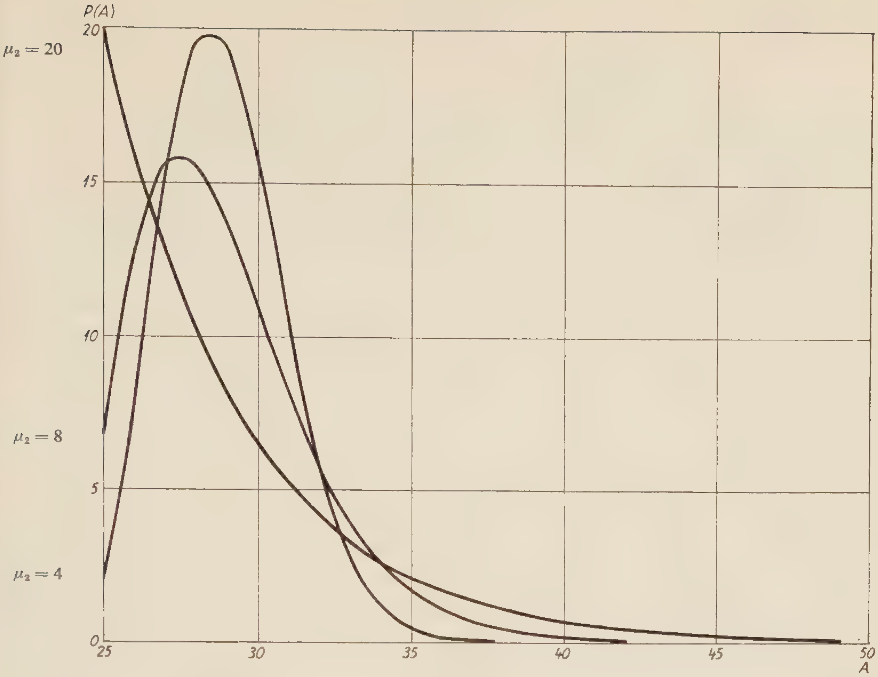

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Fig. 1. The traffic distribution

$$P(k, r, p) = \binom{r+k-1}{k} p^r \cdot q^k \qquad (k = 0, 1, 2 \ldots .)$$

for $\alpha_1 = \frac{r \cdot q}{p} = 4$ and $\mu_2 = 4,\ 8,\ 20.$

the variance

$$\mu_2 = \frac{r \cdot q}{p^2} = 5$$

and the skewness

$$\frac{\mu_3}{\mu_2{}^{3/2}} = \frac{1+q}{\sqrt{r \cdot q}} = 0.67$$

The character of the future traffic distribution can naturally not be estimated without some uncertainty. This applies not only to the mean value but also, and perhaps to an even greater extent, to the higher moments of the distribution.

Let us assume that the mean value in the traffic distribution is constant, $A_m = 29$, as in the example above, corresponding to $\alpha_1 = 4$, but that the variance changes; and let us then examine how such a change affects the number of switches. We may start from the values given in *Table 2*.

Table 2. Parameters p, k, r for constant mean value $\alpha_1 = 4$, ($A_m = 29$)

p	q	r	μ_2	μ_3	$\frac{\mu_3}{\mu_2^{3/2}}$
1	0		4	4	0.50
0.8	0.2	16	5	7.5	0.67
0.5	0.5	4	8	24	1.06
0.33 ...	0.66 ...	2	12	60	1.44
0.20	0.80	1	20	180	2.01

The values in the first row give a Poisson distribution and those in the last row a geometrical distribution.

The distributions $\mu_2 = 4$, 8 and 20 are shown in *Fig. 1*. The distribution dealt with in the example thus lies between the curves marked $\mu_2 = 4$ and $\mu_2 = 8$.

The calculations are done in precisely the same way as above (*Table 1*) and the result is shown in *Fig. 2* with $\varepsilon = \frac{b}{G}$ as function of the standard deviation of the distributions in accordance with eq. 3 for different numbers of switches. For $\varepsilon = 0.04$, as before, we read the following values for the number of switches with different standard deviations at a constant mean value $A_m = 29$.

Table 3. Number of switches with different standard deviations in the traffic distribution. Mean value constant $A_m = 29$. The table also shows the "equivalent traffic" A_{*1} and A_{*4} and the congestion $E(n, A_{*1})$.

σ	μ_2	n	A_{*4}	A_{*1}	$E(n, A_{*1})$
0	0	46	29	31.5	0.002260
2	4	48	30	33	0.002717
2.24	5	48	30	33	0.002920
2.83	8	49	31	34	0.003126
3.46	12	51	32	35	0.003333
4.47	20	~53	34	37	0.002613

Thus large changes in standard deviation in the assumed type of distribution give rise to substantial changes in the number of switches, so that it is by no means sufficient to use solely a mean value of the traffic when determining the switch requirement. If the mean value is known, however, no great measure of knowledge of the other properties of the distribution is needed. An error in assessment of the standard deviation in the example chosen of between $\sigma = 2$ and $\sigma = 2.83$ thus gives a maximum error in the number of switches of $\pm$ 1, which in view of the flatness of the minimum (*Fig. 2*, Annex 2) involves a negligible increase in the cost of switches plus the economic value of the inconvenience to subscribers due to traffic losses.

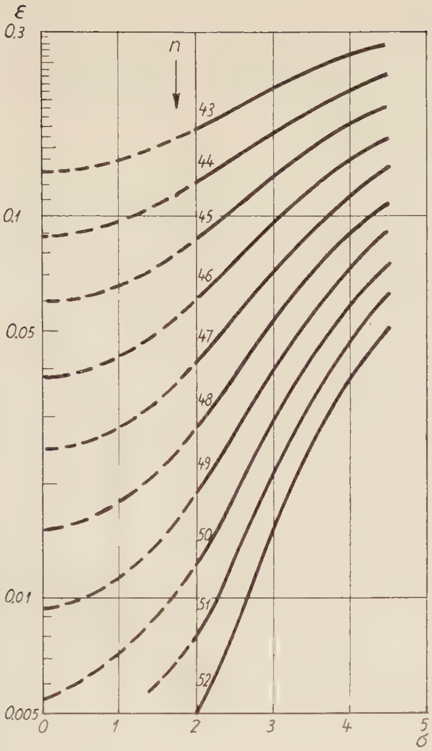

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Fig. 2. The switch provision factor $\varepsilon = \frac{b}{G}$ as function of the standard deviation σ of the distribution for different numbers of switches.

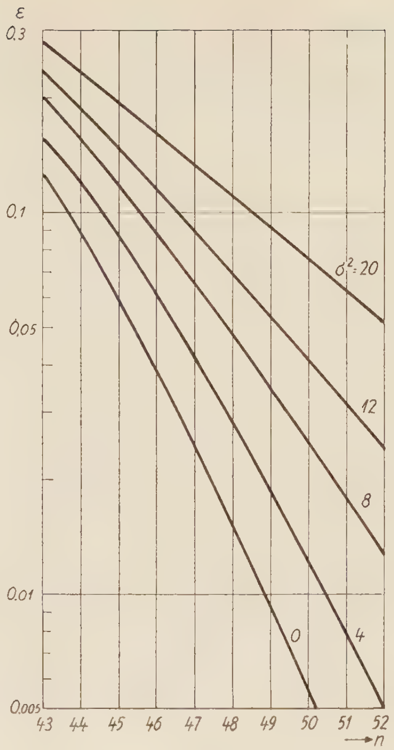

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Fig. 3. The switch provision factor $\varepsilon = \frac{b}{G}$ as function of the number of switches for different values of the variance $\mu_2 = \sigma^2$.

Fig. 3 shows instead $\varepsilon - \frac{b}{G}$ as function of n for different values of μ_2. The scale for ε is logarithmic and one observes that the number of switches in the region $0.01 < \varepsilon < 0.1$ can be written with sufficient accuracy as

$$n = c_1 + c_2 \cdot \log G$$

where c_1 and c_2 are constants dependent on μ_2.

This observation confirms that the change in the number of switches is proportional to the logarithm of the quality of service factor even if the traffic is represented by a frequency function $f(A)$, and that the proportionality factor c_2 grows with growing standard deviation in the traffic distribution (cf. Annex 2).

Finally, by way of illustration, the course of the function

$$P(A)\,[E(n, A) - E(n + 1, A)]$$

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is shown for $n = 46, 49, 52$ in *Fig. 4.* The values for $P(A)$ are taken from *Table 1.* It will be seen how, with growing n, the curve collapses and the maximum shifts to the right and how part of the traffic distribution which is determinative of the number of switches becomes increasingly smaller. Of the total contribution

$$\sum_{A=25}^{\infty} A\Delta E \cdot P(A)$$

to the improvement factor, 90 % comes at $n = 46$, 49 and 52 during periods of about 2.2, 1.8 and 1.3 hours, respectively, of the 4 hours per day described by $P(A)$.

The greater the magnitude of G, which represents the value of the subscriber's time, and the smaller the cost of a switch, *i.e.* the smaller the value of the switch provision factor $\varepsilon = \frac{b}{G}$, the higher and the rarer will be the traffic values which predominantly determine the economic number of switches on the traffic route.

In the example under consideration the switch provision is clearly determined primarily by the mean value of the traffic during a period of time in which there are appreciable traffic losses and, secondarily, by the standard deviation and skewness of the traffic distribution. The material examined, however, is too small to draw any general conclusions from it. For this purpose more comprehensive investigations would be required, which must be a matter for the future.

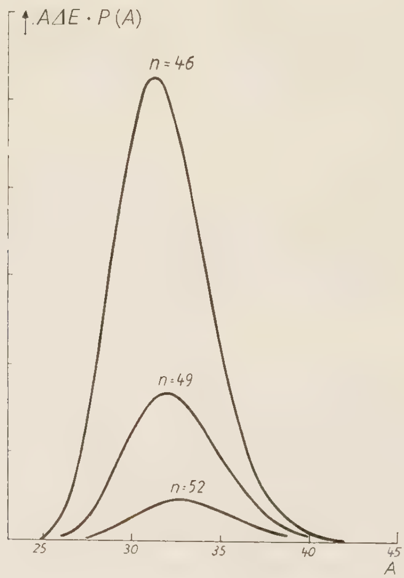

Fig. 4. $P(A)[E(n, A) - E(n+1, A)]$ as function of A for $n = 46, 59, 52$. The values of $P(A)$ are taken from Table 1.

ANNEX 4

Calculation of Number of Switches on a Route Comprising Several Selector Stages

In chapter 3 a general expression (3.12) was given for the cost of switches and the economic value of inconvenience to subscribers, *viz.*

$$K(n_1 \ldots\ n_i \ldots) = \sum_j b_j n_j + \sum_i G_i T_i \int_0^\infty A_i \cdot E_i \cdot f(A_i) \cdot dA_i \tag{1}$$

The use of this equation to determine the number of switches on routes comprising several selector stages will be illustrated in this Annex, together with a calculation for switch provision in a telephone exchange of 10,000 lines on the Ericsson 500-point selector system.

Let us initially consider a route comprising two stages in series as in *Fig. 1*, representing, for example, a transit route.

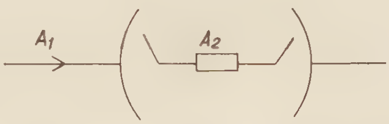

Fig. 1. Traffic route comprising two stages.

For an incoming traffic A_1 to the first selector stage comprising n_1 switches, the congestion in this stage is

$$E_1 = E_1(n_1, A_1)$$

The traffic to the second stage is

$$A_2 = A_1(1 - E_1)$$

and the congestion in the second stage with n_2 switches is

$$E_2 = E_2(n_2, A_2) = E_2(n_2, A_1(1 - E_1))$$

The sum of the traffic losses in the two stages is

$$A_1 E_1 + A_2 E_2 = A_1(1 - (1 - E_1)(1 - E_2)) = A_1(E_1 + E_2 - E_1 E_2) \tag{2}$$

An *increase* of one switch in the first stage causes a *decrease* in the traffic losses in that stage by

$$A_1 \Delta E_1 = A_1 [E_1(n_1, A_1) - E_1(n_1 + 1, A_1)] \tag{3}$$

This *decrease* in the traffic losses corresponds to an exactly equal *increase* in the traffic to the second stage, which leads to an *increase* in the traffic losses in that stage by

$$\Delta A_2 E_2 = A_{2,\, n_1+1} \cdot E_2(n_2,\, A_{2,\, n_1+1}) - A_{2,\, n_1} \cdot E_2(n_2,\, A_{2,\, n_1}) \tag{4}$$

in which the expression $A_{2,\, n_1}$ stands for

$$A_{2,\, n_1} = A_1(1 - E_1(n_1,\, A_1))$$

An increase of one switch in the first stage thus leads to a decrease in the traffic losses in both stages of

$$\Delta AE = A_1 \Delta E_1 - \Delta A_2 E_2 \tag{5}$$

An increase of one switch in the second stage causes a decrease in the traffic losses of

$$A_2 \Delta E_2 = A_2 [E_2(n_2,\, A_2) - E_2(n_2 + 1,\, A_2)] \tag{6}$$

From eq. 5 and 6 it is clear that the numbers of switches n_1 and n_2 must be determined so as simultaneously to fulfil the following conditions:

$$\varepsilon_1 = \frac{b_1}{G} \approx T \int_0^\infty \Delta AE \cdot f(A) \cdot dA \tag{7}$$

$$\varepsilon_2 = \frac{b_2}{G} \approx T \int_0^\infty A_2 \Delta E_2 \cdot f(A) \cdot dA \tag{8}$$

By similar reasoning the minimum condition for r stages in series can be written

$$\varepsilon_\nu = \frac{b_\nu}{G} \approx T \int_0^\infty A \cdot \prod_{\mu=1}^{\mu=\nu-1} (1 - E_\mu) \cdot \Delta \left[\prod_{\mu=\nu}^{\mu=r} (1 - E_\mu) \right] \cdot f(A) \cdot dA \tag{9}$$

where $A = A_1$ is the traffic entering the first stage.

From this expression, as also from (4) and (5), it is clear that

$$\boxed{\varepsilon_\nu < T \int_0^\infty A \Delta E_\nu \cdot f(A) \cdot dA} \tag{10}$$

and that, therefore, an upper limit to the number of switches in each stage can be obtained by changing the inequality sign in eq. 10 to equality sign and determining the number of switches in each stage independently.

Consideration of eq. 3 and 5 reveals that, whereas $A_1\, \Delta E_1$ for a given number of switches approaches unity for growing traffic in the same way as the expression

$$1 - \frac{1}{A} - \frac{2n-1}{A^2} - \frac{3n(n-3)+6}{A^3} - \ldots . \tag{11}$$

the expression ΔAE approaches the value

$$1 - [(n_1+1) \cdot E_2(n_2, n_1+1) - n_1 \cdot E_2(n_2, n_1)] \tag{12}$$

which may be considerably less than unity.

The minimum cost curve for several stages in series will be flatter than for a single stage. *Table 1* illustrates the character of the minimum on condition that the frequency function $f(A)$ can be represented by a single value $A_0 = 10$ valid for 1 hour and that $\varepsilon_1 = \varepsilon_2 = 0.018$.

Table 1. Cost increase per cent for increase or decrease in number of switches in two stages in series

The figures within brackets show the per cent change of cost for a single switch.

n_2 \ n_1	17	18	19	20	21	22
22	10.2	5.0	2.9	2.8	3.9	5.7
21	8.4	3.2	1.1	1.1	2.2	3.9
20	7.1	2.0	0.0	0.0	1.1	2.8
19	6.9	1.9	0.0	0.0	1.2	2.9
18	10.9	3.8	2.0	2.0	3.2	5.0
17	11.3	8.6	7.0	7.2	8.4	10.2
	(13.8)	(3.9)	(0)	(0)	(2.2)	(5.8)

Owing to the flatness of the minimum it is probable that the number of switches can be sufficiently accurately calculated by simplified methods, *e.g.* by using eq. 10. We shall return to this point later in connection with the question of plant provision for an exchange based on 500-point selectors.

Fig. 2 shows a circuit diagram for a telephone exchange built on the 500-point selector system.

The subsequent calculations are based on the following assumptions:

1. Two exactly identical exchanges with equal quantities of initiated traffic are interconnected by a junction cable with separate groups of circuits for incoming and outgoing traffic.
2. The registers in the exchanges serve all 500-line groups.

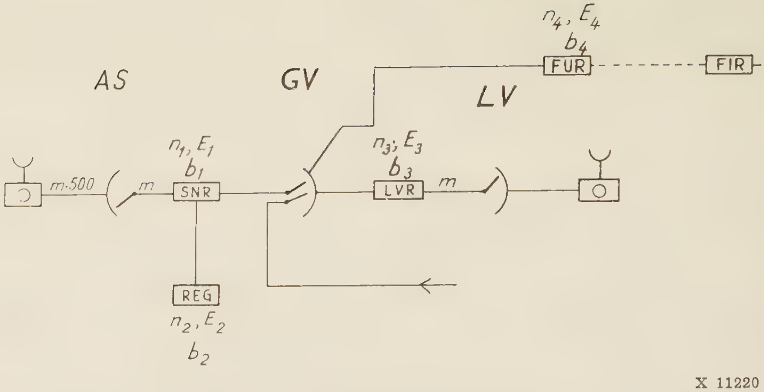

Fig. 2. Circuit diagram for a telephone exchange built on the 500-point selector system.

AS = Line Finder.
GV = Group Selector.
LV = Final Selector.
SNR = Cord Circuit Relay Set.
REG = Register.
LVR = Final Selector Relay Set.
FUR = Outgoing Junction Relay Set.
FIR = Incoming » » »

The traffic through different switches and the congestion in the exchange are calculated as follows:

A = total initiated traffic

m = number of 500-line groups

n_1 = number of SNR per 500-line group

n_2 = number of REG

n_3 = number of LVR per 500-line group

n_4 = number of FUR

$A_1 = A_{\text{SNR}} = \dfrac{A}{m}$ = traffic entering SNR

$E_1 = E_{\text{SNR}}$ = the congestion in SNR according to Erlang's formula for lost call systems

$A_2 = A_{\text{REG}} = \alpha \cdot A(1 - E_1)$ traffic entering REG

$\alpha = \dfrac{\text{mean holding time for REG}}{\text{mean holding time for SNR}}$

$E_2 = E_{\text{REG}}$ = the congestion in REG according to Erlang's formula for lost call systems

$A_3 = A_{\text{LVR}} = \dfrac{A}{m}(1 - \alpha)(1 - E_1)(1 - E_2)(1 - \eta E_4)$ = traffic entering LVR

η = initiated external traffic/total initiated traffic

$A_4 = A_{\text{FUR}} = \eta(1 - \alpha) A (1 - E_1)(1 - E_2)$ = traffic entering FUR

E_3 and E_4 are calculated from O'Dell's formulæ:

$$E = c^k$$

$$E = E_k(A_0)$$

$$A = A_0 + \frac{c}{1 - E}(n - k)$$

$$k = \text{availability} = 20$$

The total congestion in the exchange is

$$E = E_1 + (1 - E_1) \cdot E_2 + (1 - E_1)(1 - E_2)[(1 - \eta E_4) E_3 + \eta E_4]$$

or

$$E = 1 - (1 - E_1)(1 - E_2)(1 - E_3)(1 - \eta E_4)$$

The cost of the exchange plus the economic value of the traffic losses will be

$$m \cdot b_1 n_1 + b_2 n_2 + m \cdot b_3 n_3 + b_4 n_4 + GT \int_0^\infty AE \cdot f(A) \cdot dA \tag{13}$$

b_1 = the cost of each SNR kr
b_2 = the cost of each REG kr
b_3 = the cost of each LVR kr
b_4 = the cost of each FUR plus junction circuit plus FIR kr

The provision factor for the different switches will be

for SNR $\varepsilon_1 = \dfrac{b_1}{G}$

REG $\varepsilon_2 = \dfrac{b_2}{G} \cdot \alpha$

LVR $\varepsilon_3 = \dfrac{b_3}{G}(1 - \alpha)$

FUR plus junction plus FIR $\varepsilon_4 = \dfrac{b_4}{G}(1 - \alpha)$

If it is assumed that the frequency function $f(A)$ can be replaced by a single value A_*, the number of switches can be approximately determined by starting from the aforementioned plant provision factors and successively using the following traffic values:

For SNR: n_1 is determined from ε_1 and traffic $\dfrac{A_*}{m}$

For REG: n_2 is determined from ε_2 and traffic $\alpha A_*(1 - E_1)$

For FUR + junction + FIR: n_4 is determined from ε_4 and traffic $\eta(1-\alpha) A_*(1-E_1)(1-E_2)$

For LVR: n_3 is determined from ε_3 and traffic $(1 - \alpha) \cdot \dfrac{A_*}{m}(1 - E_1)(1 - E_2)(1 - \eta E_4)$

As already mentioned, an upper limit for the number of switches is obtainable on the basis of eq. 10.

Exact calculation of the numbers of switches n_1, n_2, n_3, n_4 by optimization of eq. 13 involves very considerable numerical work. The results presented here were obtained with the aid of a computer, the programme for which was worked out by S.-G. Carlsson on the following principles.

The integral in the cost function (eq. 13) cannot be explicitly determined, for which reason this expression is replaced by the approximation

$$K = m \cdot b_1 n_1 + b_2 n_2 + m \cdot b_3 n_3 + b_4 n_4 + GT \sum_A AE \cdot P(A)$$

The number of switches in any one group is changed by one switch at a time until K becomes minimum. During this search the values of the remaining groups of switches are constant. The old value of the group under consideration is replaced by the new one, and the procedure is repeated for all groups of switches. The search is done in the order n_4, n_3, n_2, n_1, and the calculation is stopped when the combination is reached which gives no change in K on continued search. The starting values are obtained by approximate calculation of the number of switches by methods already described.

The calculations are based on the following data:

Traffic distribution T = 24 hours per day.

A	$P(A)$	$F(A)$
		0.822659
345	0.012758	0.835417
375	18764	854181
405	26875	881056
435	32780	913836
465	32183	946019
495	25058	971077
525	15423	986500
555	7546	994046
585	2998	997044
615	1034	998078
645	366	998444
675	168	998612
705	101	998713
735	70	998783
765	48	998831
795	35	998866
825	25	998891
855	18	998909
885	13	998922

$\alpha = 0.1$
$m = 20$
$b_1 = 380$ kr
$b_2 = 2{,}680$ kr
$b_3 = 375$ kr
$b_4 = 1{,}430$ kr
$\eta = 0, 0.25, 0.8$
$g = 2 \quad 4 \quad 8 \quad 16 \quad 32$
$G = 9{,}125,\ 18{,}250,\ 36{,}500,\ 73{,}000,\ 146{,}000$
$r = 8\ \%$

For $\eta = 0.25$ and $b_4 = 1{,}430$ the following result is obtained for different values of g.

Table 2. Optimum number of switches n_1, n_2, n_3, n_4 for different values of G. $b_1 = 380$ kr, $b_2 = 2{,}680$ kr, $b_3 = 375$ kr, $b_4 = 1{,}430$ kr. $\eta = 0.25$.

g kr/hour	G kr	n_1	n_2	n_3	n_4
2	9,125	40	74	38	154
4	18,250	42	77	40	163
8	36,500	44	81	42	171
16	73,000	46	84	44	180
32	146,000	48	88	46	189

For $G = 36{,}500$ and different values of η the following result is obtained:

Table 3. Optimum number for switches n_1, n_2, n_3, n_4 for different values of η. $b_1 = 380$ kr, $b_2 = 2{,}680$ kr, $b_3 = 375$ kr, $b_4 = 1{,}430$ kr. $G = 36{,}500$ kr. $g = 8$ kr/hour.

	n_1	n_2	n_3	n_4
$\eta = 0$	44	81	42	—
$\eta = 0.25$	44	81	42	171
$\eta = 0.80$	44	80	42	538

Table 2 shows primarily that the number of switches changes in an arithmetical series for a change of the quality of service factor in geometrical series, which further corroborates the opinions expressed in Annex 2.

Table 3 shows that the number of switches n_1, n_2, n_3 is practically independent of η.

By means of a separate calculation one can check that a traffic of $A_* = 27.60$ during 1 hour per day leads to practically the same losses as under the assumed frequency function. Based

on this value and on the provision factors for SNR, REG and LVR on page 55, the numbers of switches n_1, n_2, n_3 will be as follows:

Table 4. Approximate number of switches n_1, n_2 n_3 calculated with aid of the "equivalent traffic".

g	n_1	n_2	n_3
2	41	75	38
4	43	78	39
8	45	81	41
16	46	84	43
32	48	86	44

The agreement with *Table 2* is striking and proves that one can use simplified methods of calculation for determining the number of switches on routes comprising several selector stages.

This simplification can be done by calculating, with the aid of the frequency function $f(A)$ as above, an equivalent traffic A_* holding good within certain limits of the number of switches or by calculating the number of switches in each stage separately in accordance with eq. 10.

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Extension of Telephone Plant with Regard to the Value of Subscribers' Time

II. Extension of Telephone Plant to Match a Growing Need

BY

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In a previous article, ***Economic Optimum for Conductor Diameter and Number of Switches on Telephone Circuits***, one of the questions considered was that of switch provision on a traffic route on the assumption that the number of subscribers and the frequency function $f(A)$ for the traffic underwent no change during the period under consideration.

In the present paper we shall abandon this premise and instead assume that the need for telephones and the traffic through different switches and groups of circuits grow with time. From this point of departure we shall attempt to find means for deciding at what time and in what quantities the plant should be added to.

On a route carrying a traffic load that continues to grow with time, there arise losses which increase very rapidly with the load. These losses represent a growing inconvenience to subscribers, the economic value of which ultimately attains a level at which expansion of the plant becomes necessary.

In a telephone exchange to which increasing numbers of subscribers are connected, there is a risk that a certain number of potential subscribers cannot obtain a telephone immediately but must wait until the exchange is extended.

The inconveniences incurred by subscribers in both these cases can be converted into money terms by use of the quality of service factor G discussed in the previous paper; and the time and the size of future extensions of the plant must be determined so that the sum of the present value of anticipated plant costs and inconveniences is as small as possible.

Paper to be presented at The Third International Teletraffic Congress in Paris, September 11—16, 1961.

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Contents

Page

CHAPTER 1 *General Principles for Determining the Time and the Size of Plant Extensions Having Regard to the Value of Subscribers' Time* 63

1.1 Calculation of size of instalments without regard for the inconvenience to subscribers 67

1.2 Calculation of times of instalments on assumption that the capacity of each instalment is already fixed 68

CHAPTER 2 *What Requirements Should be Placed on Prognoses of Future Growth?* . . 69

CHAPTER 3 *Calculation of Size of Plant Extensions When the Growing Need is Defined by a Frequency Function* 75

CHAPTER 4 *Estimate of Size Plant Extensions Having Regard to the Risk of Stagnation in the Need* 81

CHAPTER 5 *Choice Between Extension in One and Extension in Two Stages* 84

CHAPTER 6 *Extension of Plant Serving a Growing Number of Subscribers* 87

CHAPTER 7 *Extension of Plant to Provide for Growing Traffic* 94

ACKNOWLEDGEMENTS 101

CHAPTER 1

General Principles for Determining the Time and the Size of Plant Extensions Having Regard to the Value of Subscribers' Time

The need for telephones and the traffic through different switches and groups of circuits generally grow with time. Therefore, when calculating the capacity for which a plant should be built, it is not sufficient to consider the immediate requirements alone. Attention must also be paid to future extensions and to their probable cost.

Nothing definite can be said about the character and magnitude of a growing need, even when predicted with the utmost care. But it is usually possible to fix certain limits within which the need may be expected to lie with some probability. A reasonable but naturally by no means sure method of estimating such limits is *ex post facto* to extrapolate observed growth curves and to compare the result with the real historical development.

When planning a plant of a given type an answer is required to two questions, *viz.*

1. How large a capacity must the plant have?
2. When must it be replaced or extended?

Insofar as the cost of a plant is directly proportional to the number of installed units, it is clearly most economical to instal one unit at a time at periods consistent with the growth of the need and with the inconveniences which become greater the greater the need. The questions arising in conjunction with a plant cost structure of this kind have already been dealt with in the paper, *Economic Optimum for Conductor Diameter and Number of Switches on Telephone Circuits.*

The first question therefore arises only in conjunction with the installation of a group consisting of a number of indivisible switching units, the cost of which per unit diminishes with the number of units in the group. The answer to this question is predominantly determined by the magnitude of the future growth and by the extent to which the cost per switching unit diminishes with growing number of units.

The answer to the second question is determined predominantly by the cost of the inconveniences arising if the line capacity or traffic capacity of the plant is insufficient, by the future cost of installation of a new plant, and by the maintenance cost. These costs are in turn dependent on the capacity of the plant. Manifestly, therefore, the capacity of a plant and the time of its installation are to some extent interdependent and should therefore be decided simultaneously.

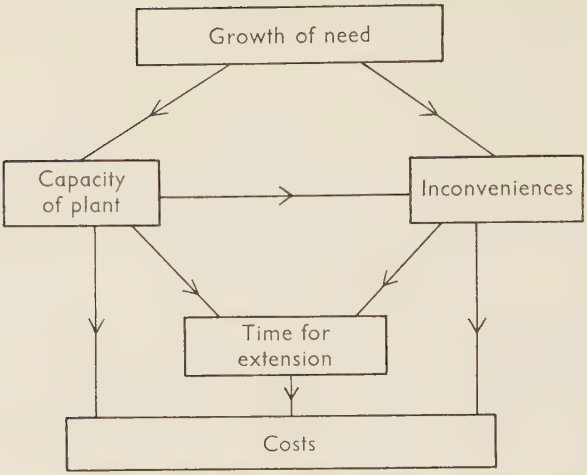

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Fig. 1.1. Interdependence between growth of need, capacity of plant, inconveniences to subscribers, time for extension, and costs.

The relation between costs and cost-determining factors is illustrated in *Fig. 1.1.*

Following thereon, it will be shown how the time and size of a future extension of already existing plant of a certain capacity can be simultaneously determined on condition of a continuously growing need. This method of presentation will form the basis for the investigations that follow. On account of the nature of the problems it will be possible to make certain simplified assumptions, which will facilitate the calculations without loss of accuracy.

The assumptions for this study are the following:

n_0 = the total, in some cases unutilized, capacity of the plant at time $t = 0$ when the study is made.

s = the need at time t years, or in some cases the growth of the need during the period $0 - t$.

$\varphi(s, t) \cdot ds$ = the probability of the need at time t being between s and $s + ds$

$n_1, n_2 \ldots$ = the capacity of future extensions installed at times $t_1, t_2 \ldots$.

ψ_ν = $\psi(s, n_0 + n_1 + \ldots + n_\nu)$ = the economic value of the intensity of inconveniences to subscribers for a need s after ν extensions have been installed

$$\Psi_\nu = \Psi(n_0 + n_1 + \ldots + n_\nu, t) = \int_0^\infty \psi_\nu \cdot \varphi_\nu(s, t) \cdot ds$$

i.e. the expected economic value of the intensity of inconveniences at time t. This expression holds good for $t_\nu < t < t_{\nu+1}$.

In the expression above for Ψ_ν, $\varphi(s, t)$ has been replaced by $\varphi_\nu(s, t)$. This has been done in order to indicate that the frequency function $\varphi(s, t)$ may differ in character at different times, as may happen, for example, if one adopts the hypothesis that one's knowledge of the future is not the same as at $t = 0$, but increases with the passage of time. In such case, if s denotes the growth of the need, it may happen that

$$\varphi_\nu(s, t) = \varphi(s, t - t_\nu)$$

K_ν $= K(n_\nu) =$ the cost of an extension consisting of n_ν switches (or lines). This cost is assumed to be independent of the time at which the extension is made.

On the basis of these assumptions we may immediately set up the following expression for N_0 representing the present value of all costs of extensions including the expected economic value of inconveniences:

$$N_0 = K_1 \cdot e^{-pt_1} + K_2 \cdot e^{-pt_2} + \ldots + \int_0^{t_1} \Psi_0 \cdot e^{-pt} \cdot dt + \int_{t_1}^{t_2} \Psi_1 \cdot e^{-pt} \cdot dt + \ldots \tag{1.1}$$

The times $t_1, t_2 \ldots$, and the size of extensions $n_1, n_2 \ldots$, must be determined so that eq. 1.1 is a minimum. A necessary condition for this is that

$$\frac{\partial N_0}{\partial t_i} = 0 \qquad \frac{\partial N_0}{\partial n_i} = 0 \qquad (i = 1, 2, \ldots) \tag{1.2}$$

with the obvious constraint

$$0 < t_1 < t_2 < \ldots\ldots\ldots\ldots < t_\nu < \ldots \tag{1.3}$$

Written in full the system of equations (1.2) reads

$$\left.\begin{array}{l}
\Psi(n_0, t_1) - \Psi(n_0 + n_1, t_1) = p \cdot K(n_1) \\
\Psi(n_0 + n_1, t_2) - \Psi(n_0 + n_1 + n_2, t_2) = p \cdot K(n_2) \\
\text{— — — — — —} \\
\displaystyle\int_{t_1}^{t_2} \frac{\partial \Psi_1}{\partial n_1} e^{-pt} \cdot dt + \int_{t_2}^{t_3} \frac{\partial \Psi_2}{\partial n_1} e^{-pt} \cdot dt + \ldots + \frac{\partial K_1}{\partial n_1} e^{-pt_1} = 0 \\
\displaystyle\int_{t_2}^{t_3} \frac{\partial \Psi_2}{\partial n_2} e^{-pt} \cdot dt + \ldots + \frac{\partial K_2}{\partial n_2} e^{-pt_2} = 0 \\
\text{— — — — — —}
\end{array}\right\} \tag{1.4}$$

In these expressions $p = \ln(1 + r)$

$r =$ the annual interest factor

At first sight this seems rather intimidating. But a simplification can be immediately achieved if it is observed that

$$\frac{\partial \Psi_\nu}{\partial n_i} = \frac{\partial \Psi_\nu}{\partial n_j} \qquad (i, j = 1, 2 \ldots \nu)$$

A further simplification is possible if one assumes that the cost K of an extension is a linear function of the capacity n. This is usually the case in reality, at least for a given interval

$$n_{\min} < n < n_{\max}$$

One can then put

$$K_i = a + b \cdot n_i \qquad (i = 1, 2 \ldots)$$

where a and b are constants. By subtraction of the equations in the last group of 1.4 two by two, and suitable change in the order of sequence, the system of equations can be written in a more attractive manner:

$$\left.\begin{array}{l} \Psi(n_0, t_1) - \Psi(n_0 + n_1, t_1) = p\,(a + bn_1) \\ \quad - \displaystyle\int_{t_1}^{t_2} \frac{\partial \Psi_1}{\partial n_1} \cdot e^{-pt} \cdot dt = b\,(e^{-pt_1} - e^{-pt_2}) \\ \Psi(n_0 + n_1, t_2) - \Psi\,(n_0 + n_1 + n_2, t_2) = p\,(a + bn_2) \\ \quad - \displaystyle\int_{t_2}^{t_3} \frac{\partial \Psi_2}{\partial n_2} \cdot e^{-pt} \cdot dt = b\,(e^{-pt_2} - e^{-pt_3}) \\ - - - - - - \end{array}\right\} \tag{1.5}$$

From this a simple means of solving this system of equations is immediately apparent.

As will be seen, the first equation gives a relationship between n_1 and t_1, the second between t_1 and t_2, the third between t_2 and n_2, the fourth between n_2 and t_3, etc. This means that the calculations require only one parameter, *e.g.* n_1. But the calculation must end somewhere! This can be arranged by inserting, in place of the cost $a + bn_j$ for a given extension j, a value N_j representing the estimated value of future costs up to time t_j, or alternatively, by delimiting a given period $0 - t_h$ and setting t_1 and n_1 so that the costs during that period are as low as possible.

From the system of equations 1.4, it is apparent that the very first equation

$$\Psi\,(n_0, t_1) - \Psi\,(n_0 + n_1, t_1) = p \cdot K\,(n_1)$$

gives a relation between the magnitudes n_1 and t_1, which are of interest at the time of calculation. Other instalments n_2, n_3 . . . and the times for them will only be of interest later on when the growth can be better predicted. At the time of calculation $t = 0$ these magnitudes are merely a mathematical aid, serving as boundary conditions for the relation between n_1 and t_1 in the expression above.

In the continuation of this chapter the problem will be simplified in the respect that, on the one hand, we shall calculate the size of instalments without regard for the inconveniences and, on the other, that we establish the time for an extension on the assumption that its size is already decided. In addition to the greater lucidity of this approach, it provides approximate methods for determining the time and size of future extensions, which will almost always be sufficiently accurate for practical purposes.

1.1 Calculation of size of instalments without regard for the inc onvenience to subscribers

Let the cost of a plant be represented by the linear cost function

$$a + b \cdot n$$

which is sufficiently accurate at least for an interval $n' < n < n''$ and in which a and b are constants.

Let the need at time $t = 0$ be n_0 and at time t years

$$n_0 + ct$$

where c is the growth of the need per annum.

On these assumptions the cost of the first extension, designed to cover the requirement during a period t_x, is clearly

$$a + b(n_0 + ct_x) \tag{1.6}$$

Denote by N the value of all installations from time t_x reduced to this point of time by the use of compound interest. The present value of all outlays related to time $t = -0$ is then

$$N_0 = a + b(n_0 + ct_x) + N \cdot e^{-pt_x} \tag{1.7}$$

Minimization of N_0 with respect to t_x gives

$$N = \frac{bc}{p} \cdot e^{pt_x} \tag{1.8}$$

N is dependent on the growth after time t_x, on the cost of the plant installed at that time, and on the anticipated future extensions. As soon as N has been estimated[1], one can obviously determine t_x from eq. 1.8; the size of the first extension is then $n_0 + ct_x$ and its cost is evaluated according to eq. 1.6. A simple means of estimating N, which is usually sufficiently accurate, is to assume that the need continues to grow in the same way and that all equipment installed is of the same kind.

On these assumptions, as can be easily proved[1], the instalments as from time t_x will be of equal size.

---

[1] If the calculation comprises several instalments, so that

$$N_0 = \sum_{\nu=0}^{m-1} [a + bc\,(t_{\nu+1} - t_\nu)]\; e^{-pt_\nu} + N \cdot e^{-pt_m}$$

the economic period of provision t_1 for the first instalment is obtainable from

$$pt_1 = \ln\left\{1 + pt_0 + \ln\left[1 + pt_0 + \ldots + \ln\left(1 + pt_0 + \ln\frac{N}{\frac{bc}{p}}\right)\ldots\right]\right\}$$

with
$$t_0 = \frac{a}{bc}$$

The greater the number of instalments included in the calculation, the less claim for accuracy can be made in the estimate of N. If $m \to \infty$, the determination of the economic period of provision for all instalments will be governed by

$$pt_x = \ln(1 + pt_0 + pt_x) \tag{cf. 1.10}$$

The present value of N at time t_x is then determined from the equation

$$N = a + bct_x + N \cdot e^{-pt_x} \tag{1.9}$$

which, with the general minimum condition (1.8), gives

$$p\,(t_0 + t_x) = e^{pt_x} - 1 \tag{1.10}$$

where

$$t_0 = \frac{a}{bc}$$

Eq. 1.10 determines t_x, hence the size and cost of the first extension can be calculated.

1.2 Calculation of times of instalments on assumption that the capacity of each instalment is already fixed

A plant of a given capacity is already in operation. The continuously growing need gives rise to growing inconvenience; and having regard to the economic value of this inconvenience, one wishes to determine the time at which an extension should be made. The size of the extension is assumed to have been determined either through calculations according to 1 above or by selection of a standard design of equipment. This means that the costs $K_1, K_2 \ldots$ in eq. 1.2—1.5 are constants and that the inconvenience functions $\Psi_0, \Psi_1, \Psi_2 \ldots$ can be expressed as functions solely of time t. The times $t_1, t_2 \ldots$ for the installations will then be determined from eq. 1.4.

$$\begin{aligned} \Psi_0(t_1) - \Psi_1(t_1) &= p \cdot K_1 \\ \Psi_1(t_2) - \Psi_2(t_2) &= p \cdot K_2 \\ - - - &- - - \end{aligned} \tag{1.11}$$

It should be pointed out in this context that the costs $K_1, K_2 \ldots$ are capitalized plant costs, calculated by multiplying the first cost by a present value factor

$$\mu = 1 + \frac{1 - s}{(1 + r)^{t_a} - 1} + \frac{u}{r}$$

where

r = interest factor

t_a = utilization time of the plant, years

s = difference between scrap value of dismantled plant and dismantling cost in relation to first cost

u = the ratio of the mean annual cost of maintenance to first cost.

What Requirements Should be Placed on Prognoses of Future Growth?

In plans aimed at matching plant expansion to a need which grows with time it is important to make every effort to obtain as reliable a prognosis as possible.

Certainty about future developments can never be reached, however much work and money is expended on their study. There must consequently be an upper limit to the effort that should be sacrificed on prognoses.

For this very reason it is interesting to study how the cost of a series of plant extensions, matched to a growing need, varies under different assumptions concerning the growth of the need.

An investigation of this kind has a significance also when it comes to choosing a model which describes the future development with sufficient accuracy.

Another desideratum for this model is that it shall be as simple as possible, in order that the calculations shall not be unduly comprehensive and the result difficult to evaluate.

In considering a growing need there are, of course, several models to choose from. One method would be to assume that the need n at time t years has a probability

$$P(n, t) = \frac{(ct)^n}{n!} \cdot e^{-ct} \qquad (2.1)$$

according to Poisson's formula. Thus, in this formula, c represents the trend, *i.e.* the anticipated mean increase per annum. This model has the advantage that, theoretically, it fairly well describes the process according to which the need increases with time. Its use for the calculations sketched in chapter 1, however, leads to complicated expressions.

Before accepting such a point of departure, therefore, one should consider whether a simpler model would not adequately describe the growing need and yield usable results.

In the sequel it will be assumed, initially, that the growing need can be described by a single value, c.

The cost of a plant that is extended at regular intervals of time t is

$$a + bct + (a + bct) \cdot e^{-pt} + (a + bct)\, e^{-2pt} + \ldots\ldots\ldots$$

or proportional to

$$y = \frac{t + t_0}{1 - e^{-pt}} \qquad (2.1)$$

where

$$t_0 = \frac{a}{bc}$$

This expression attains a minimum value, y_1, at a value of $t = t_1$ which satisfies eq. 1.10. If eq. 2.1 is developed around the minimum point (y_1, t_1), and taking into account that $\frac{dy}{dt} = 0$,

$$y = y_1 + \frac{1}{2} \frac{p}{1 - e^{-pt_1}} (t - t_1)^2 + \ldots . \tag{2.2}$$

and the addition Δpt_1 which arises for a relative cost increase $\frac{\Delta y_1}{y_1}$ will then be

$$\Delta pt_1 \simeq \sqrt{2\left(\frac{\Delta y_1}{y_1}\right)(e^{pt_1} - 1)} \tag{2.3}$$

This relationship gives a good approximate picture of the flatness of the minimum. Eq. 1.10 is represented in *Figs. 2.1* and *2.2*. The economic period of provision t at a known rate of interest p is obtained from these diagrams by dividing the read value of pt by p. The dotted curve in the diagrams indicates an approximate relationship between pt and pt_0, *viz.*

$$pt \cong \sqrt{2pt_0} \tag{2.4}$$

which, however, only gives a satisfactory approximation for small values of pt_0.[1]

The curves marked 0.5, 1.0 and 2 % indicate the limits within which the economic period of provision may be chosen without the cost exceeding the minimum by amounts greater than these percentages.

The flatness of the minimum is striking. At $pt_0 = 0.1$, for example, $pt = 0.42$, the optimal value. But pt can be increased to 0.52 and diminished to 0.32 without the cost increasing by more than 1 per cent. This corresponds to a variation in the capacity of plant instalments by nearly 25 per cent. At a growing value of pt_0, *i.e.* when the fixed costs a are high compared with the variable costs bc, the interval diminishes. Thus, for example, at $pt_0 = 1.0$, which

[1] An equation identical to 2.4 is often used for determining the economic size of an order. In this case one must obviously put

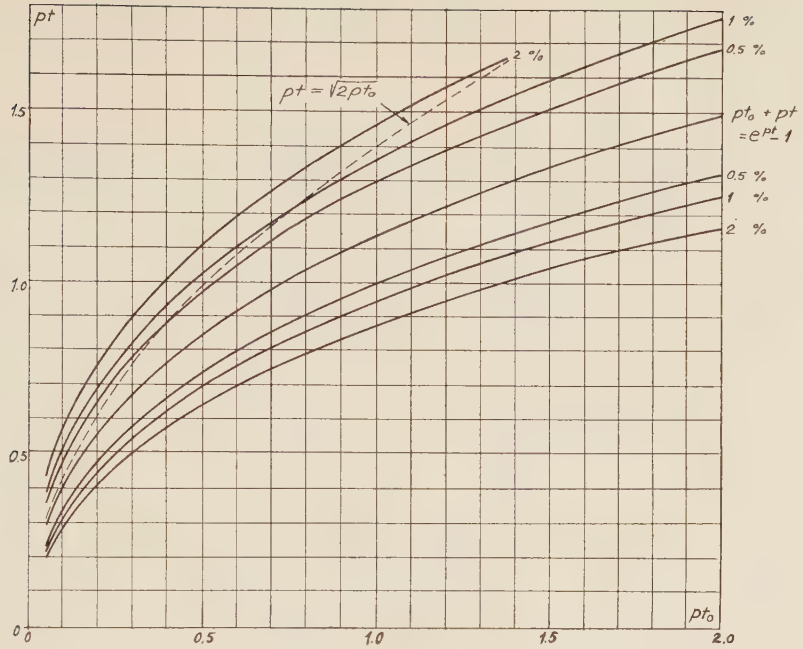
$a =$ the cost of an order irrespective of its size
$l = (p \cdot b) =$ the storage cost per unit and annum
$c =$ consumption per annum.

The economic size Q of the order will then be, according to eq. 2.4,

$$Q = \sqrt{\frac{2ac}{l}}$$

The same formula can be used to determine the intervals for installation of telephone exchanges. In this case

$a =$ fixed costs of an installation such as travelling expenses, administration cost, etc.
$b =$ installation cost per line
$c =$ growth of requirement per annum



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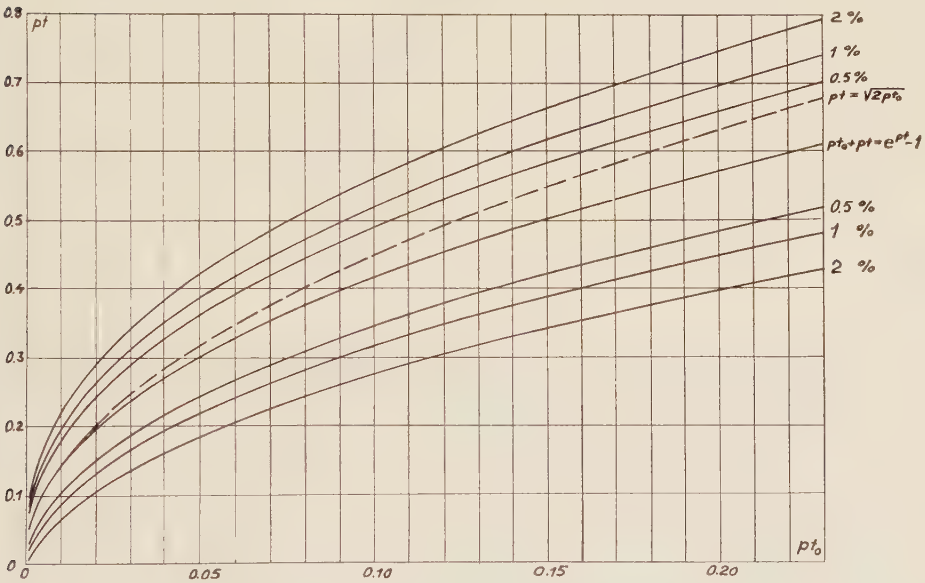

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Fig. 2.1 and 2.2. pt as function of $pt_0 = p \cdot \frac{a}{bc}$

$p = \ln(1+r)$
r = the annual interest factor
a = cost of plant independent of size
b = cost of plant per extended unit
c = growth of the need per annum.

corresponds to an economic period of provision of about 14 years, it is possible to vary the capacity of each instalment by about 18 per cent without the cost varying by more than 1 per cent.

The above remarks illustrate the flatness of the minimum for a variation in capacity of instalments with constant increase in the number of subscribers.

To illustrate how the cost varies under different assumptions as regards the magnitude of growth of the need, we assume that, within a given interval of time $t - h$ to $t + h$, one may anticipate with equally great probability at every point of time τ that an investment equal to unity will be required.

Under this assumption the anticipated present value of the investment will be proportional to

$$\varkappa = \int_{t-h}^{t+h} e^{-p\tau} \cdot \frac{1}{2h} \cdot d\tau = \frac{\sinh(ph)}{(ph)} \cdot e^{-pt} \qquad (2.5)$$

The value of $\varkappa \cdot e^{pt}$ for different values of ph is tabulated below:

ph	$\varkappa \cdot e^{pt}$
0	1
0.1	1.0017
0.2	1.0053
0.3	1.0151
0.4	1.0269
0.5	1.0422
0.6	1.0611
0.7	1.0837
0.8	1.1101
0.9	1.1405
1.0	1.1752

If one were to assume that an investment would become necessary at a time which could with equal probability assume any value whatsoever between 6.7 and 13.3 years, corresponding to an uncertainty in assessment of the future growth in the ratio 1:2, then the anticipated present value of the investment would exceed by about 1 per cent the value resulting were the time to assume a mean value of 10 years.

The above shows clearly that one should be able to employ a simple model to describe the character of the future growth of the need and should often be able to make do with a single value c representing the anticipated mean growth of the need per annum.

From this point of departure we shall hereafter examine the requirements which should be placed on the accuracy of prognoses in order that, on any deviation between the estimated and the actual growth of the need, the cost shall not exceed a predetermined value.

Assume that the size of an extension has been decided on the basis of a growth c_1 and that the plant has been planned and built to a capacity $c_1 t_1$ calculated with the aid of, for example, eq. 1.10.

After the plant has been built, the actual growth is found to deviate from the estimate and to be c_2 instead of c_1.

The present value of the cost will then be N_{12} instead of N_1 as calculated:

$$N_1 = \frac{bc_1}{p} \cdot e^{pt_1}; \tag{2.6}$$

$$N_{12} = \frac{a + bc_1 t_1}{1 - e^{-pt_1 \cdot \frac{c_1}{c_2}}} = \frac{bc_1}{p} \cdot \frac{e^{pt_1} - 1}{1 - e^{-pt_1 \cdot \frac{c_1}{c_2}}}; \tag{2.7}$$

If one could have foreseen from the outset that the growth would have been c_2 instead of c_1, the plant should have been planned for another economic period of provision t_2 and another capacity $c_2 t_2$. The present value of the cost would in such case have been

$$N_2 = \frac{a + bc_2 t_2}{1 - e^{-pt_2}} = \frac{bc_2}{p} \cdot e^{pt_2}; \tag{2.8}$$

Since the prognosis was incorrect, the plant has been more expensive. The relative increase is represented by the expression

$$\frac{N_{12} - N_2}{N_2} \tag{2.9}$$

From the above expressions the relationship can be calculated between the actual and the estimated growth of the need as function, for example, of the planned economic period of provision t_1 under different assumptions concerning the limit within which the cost increase should lie. This is illustrated in *Figs. 2.3* and *2.4*.

These diagrams show that the requirements to be placed on the accuracy of the prognosis are not too great. As an example we may assume a planned economic period of provision of $pt_1 = 0.5$, which at 8 per cent interest corresponds to 6.25 years. As appears from *Fig.2.3*, the real growth may then be 24 per cent higher or 38 per cent lower than the prognosis without the long-term cost increasing by more than 1 per cent. This corresponds to a width in evaluation of the development corresponding to $\frac{1.24}{0.62}$ or 2:1—a level of accuracy which should generally be attainable.

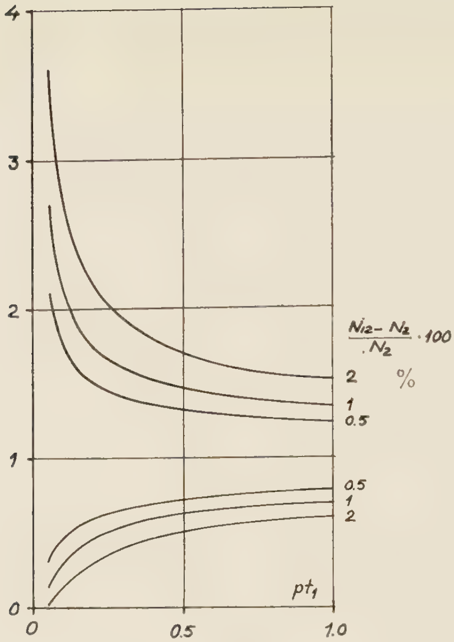

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Fig. 2.3. Requirements on the accuracy of prognosis expressed as $\frac{c_1}{c_2} = \frac{\text{estimated growth}}{\text{observed growth}}$ as function of the period of provision $\times p = pt_1$ calculated according to the estimated growth c_1 without the relative increase in costs being greater than 0.5, 1, and 2 %.

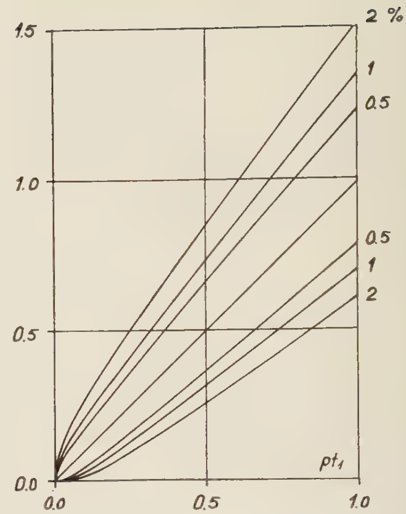

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Fig. 2.4. Permissible error in the forecast of the economic period of provision without the relative increase in costs being greater than 0.5, 1, and 2 %.

Figs. 2.3 and *2.4* also show that the requirements of accuracy in the prognosis become greater the longer the period for which the plant is planned. This demand is clearly contrary to what is attainable in reality, since the possibility of even roughly foreseeing a development becomes very much smaller the longer the period of the prognosis.

Thus at $pt = 1.0$, corresponding to an economic period of provision of 12.5 years at 8 per cent interest, an uncertainty in the subscriber growth in the ratio 2:1 corresponds to a risk of a rise in cost by nearly 2 per cent.

With the exception of underground conduits, armoured cables and buildings, plant need not generally be planned for a longer period than 10 years, and the demands of accuracy in the prognosis are therefore generally fairly small.

And, of course, as time progresses, there is all the better chance of reassessing the development and correcting errors in the last prognosis. In this way the cost of faulty prognoses can to some extent be reduced.

But this should not lead one to believe that efforts devoted to prognoses of the capacity required for different parts of a telephone plant are not worth while. Any savings in plant which can be achieved without being swallowed up in the cost of the prognosis are, of course, always welcome.

Then, again, the requirements in respect of different parts of a plant vary greatly in character. For example, the need that may be expected in already built-up areas is generally very slight and has a tendency to slow up and to become saturated, whereas in expanding areas it continues to grow without tendency to saturation within a predictable future. Therefore, plans can by no means be based on an average requirement for the plant as a whole. Admittedly, as we have seen, the cost minimum is very flat and permits some scope for standard designs of equipment and subjective assessment of the requirements. But if one departs too far from the economic optimum, the cost rises rapidly.

Even if, in the long run, the cost does not greatly change as a result of a moderate increase or decrease in the size of instalments, the cost of individual instalments changes greatly with their capacity. For this reason, and in view of the generally scarce supply of capital, it is important to devote care to the prognosis.

If a study of the problem shows that the need within an area has a tendency to stagnate, the question of the most suitable capacity of expansion will be a matter of balancing the immediate savings obtainable by installing a smaller capacity against the possible future increase in cost should the need continue to grow. This question is dealt with in chapter 4.

CHAPTER 3

Calculation of Size of Plant Extensions When the Growing Need is Defined by a Frequency Function

It has been shown in the preceding account that the stages of expansion may vary within fairly wide limits without any appreciable change in their costs. This suggests that it should not be necessary to employ a complicated model to describe the growing need, but that a very simple model should be adequate.

In the sequel, therefore, we shall investigate the simple hypothesis that the increase in the need, c, can be represented by a frequency function $f(c)$, *i.e.* that the need follows a straight line with a given probability density.

Admittedly experience has shown that a growing need is generally exponential, but there are nevertheless strong reasons for basing the calculation on a linear increase, *viz.*

1. An exponential rate of growth is, of course, a reasonable assumption which accords with earlier experience, but it is no more than an assumption. There are even advocates of an entirely contrary opinion, namely that the future rate of growth will diminish.

2. If an assumed exponential rate of growth is greater than the rate of interest, the present value of future extensions will grow with time. This means that too much importance would be attached to future events, which are more difficult to foresee the further off in time they are from the time of calculation.

3. Economic optima in this context are flat, so that no other assumptions concerning the character of the growth would seem to be necessary than that it lies within a sector delimited by two straight lines.

Under these assumptions the economic stages of expansion can be calculated on two fundamentally different assumptions.

In the first place we may assume that our knowledge of the future does not change with time, *i.e.* that the growth will follow the straight line ct with a probability density $f(c)$.

If all extensions are equally large and t is the period of provision for a growth corresponding to the mean growth c_m, the cost of one instalment will obviously be

$$a + bc_m \cdot t \tag{3.1}$$

For a growth c the period of provision (*Fig. 3.1*) will be

$$t \cdot \frac{c_m}{c} \tag{3.2}$$

If N_1 denotes the expected present value of all costs for extension at time $t = 0$, obviously

$$N_1 = (a + bc_m t) \cdot \int_0^\infty \frac{f(c) \cdot dc}{1 - e^{-pt\frac{c_m}{c}}}; \tag{3.3}$$

From this expression it is possible to determine the size of the extension, $c_m \cdot t$, by establishing the minimum of N_1.

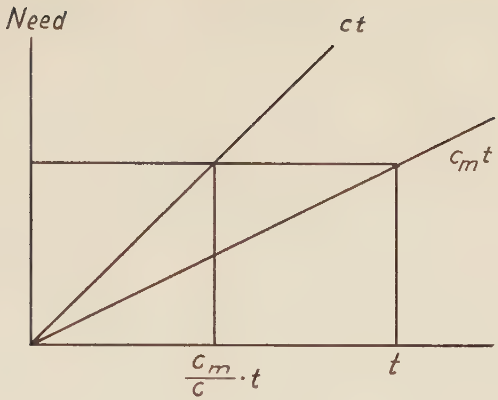

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Fig. 3.1. Period of provision for an extension of capacity $c_m \cdot t$ on condition that the growth of the need follows the straight line ct with the probability $f(c)$.

In the second place we may assume that our knowledge of the future increases with time so that, at a given point of time, τ, we consider we know that the development will follow the line

$$c(t-\tau) \tag{3.4}$$

with the probability density $f(c)$: see *Fig. 3.2.*

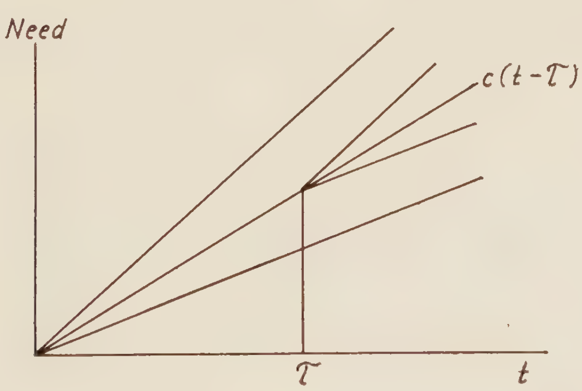

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Fig. 3.2. The growth of the future need on condition that the development follows the line $c(t-\tau)$ with the probability $f(c)$.

This means that the present value of all extensions, N_2, related to an arbitrary time of extension, is constant.

The expected present value of the first extension is, as before,

$$a + bc_m \cdot t \tag{3.5}$$

For determination of N_2 we thus obtain the equation

$$N_2 = a + bc_m \cdot t + N_2 \int_0^{\infty} e^{-pt \cdot \frac{c_m}{c}} \cdot f(c) \cdot dc\,; \tag{3.6}$$

whence

$$N_2 = \frac{a + bc_m\, t}{1 - \int_0^{\infty} e^{-pt \cdot \frac{c_m}{c}} \cdot f(c) \cdot dc}\,; \tag{3.7}$$

For one and the same frequency function, eq. 3.7 naturally gives a lower value for the economic size of instalments than eq. 3.3.

Calculations with the aid of eq. 3.3 and 3.7 can be made for different types of frequency function. Such calculations are, however, fairly laborious unless very simple hypotheses are chosen for the frequency function. Calculations based on the following simple assumptions concerning the frequency function $f(c)$ indicate that it would hardly be worth the trouble to employ more complicated hypotheses.

1. *Maximum hypothesis.* The process is assumed to follow either the straight line $c_1 t$ or the straight line $c_2 t$ with equal probability.

2. *Minimum hypothesis.* The process is assumed to follow the straight line ct with the probability

$$f(c) = \frac{c_1 c_2}{c_2 - c_1} \cdot \frac{1}{c^2} \qquad (c_1 > c > c_2) \tag{3.8}$$

This means that a small increase in the need is more probable than a large. Since the need follows the straight line $y = ct$, this expression can also be written

$$f(t) = \frac{1}{t_2 - t_1} \tag{3.9}$$

signifying that the probability of an extension being required is constant within the time interval t_1—t_2 determined by the points of intersection between a line parallel with the t-axis and the straight lines $c_2 t$ and $c_1 t$, respectively (*Fig. 3.3*).

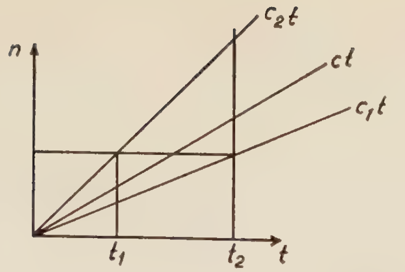

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Fig. 3.3. Transformation of the frequency function $f(c) = \frac{c_1 c_2}{c_2 - c_1} \cdot \frac{1}{c^2}$

$$f(c) \cdot dc = \frac{c_1 c_2}{c_2 - c_1} \cdot \frac{1}{c^2} \cdot dc = \frac{c_1 c_2}{c_2 - c_1} \cdot \frac{1}{\left(\frac{n}{t}\right)^2} \cdot - n \cdot \frac{dt}{t^2} = - \frac{1}{\frac{n}{c_1} - \frac{n}{c_2}} \cdot dt = \frac{1}{t_2 - t_1} \cdot dt.$$

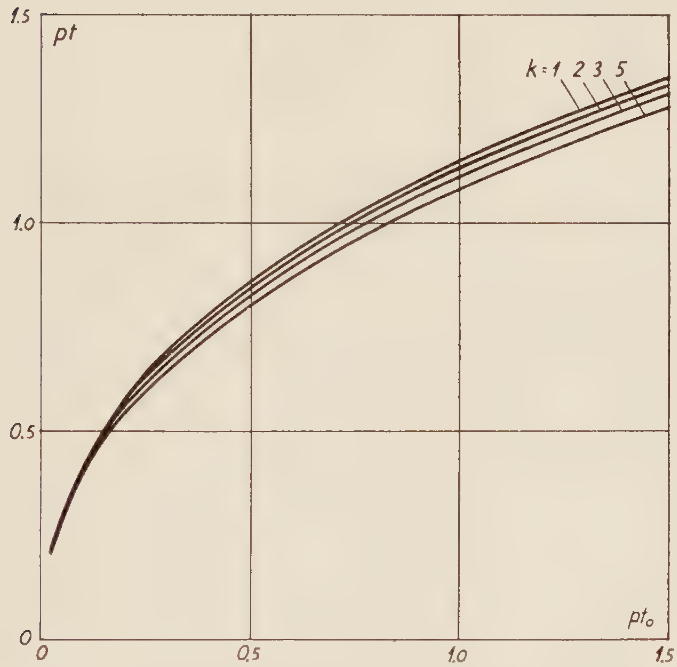

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Fig. 3.4. Continuous yearly rate of interest economic period of provision $= pt$ as function of $pt_0 = p \cdot \frac{a}{b \cdot c_m}$ for different values of $k = \frac{c_2}{c_1}$ under the "maximum hypothesis".

$$p(t + t_0) = 2 \cdot \frac{\chi_1 + \chi_2}{(1 + k) \cdot e^{-pt_1} \cdot \chi_1^2 + \left(1 + \frac{1}{k}\right) \cdot e^{-pt_2} \cdot \chi_2^2}$$

$$p_1 = \frac{p}{2}(1 + k)$$

$$p_2 = \frac{p}{2}\left(1 + \frac{1}{k}\right)$$

$$\chi_1 = \frac{1}{1 - e^{-p_1 t}}$$

$$\chi_2 = \frac{1}{1 - e^{-p_2 t}}$$

X 11230

Fig. 3.5. pt as function of $pt_0 = p \cdot \dfrac{a}{b \cdot c_m}$ for different values of $k = \dfrac{c_2}{c_1}$ under the "minimum hypothesis".

$$p\,(t_{01} + t_1) = \frac{e^{pt_1} - \omega}{\omega - \dfrac{1}{pt_1} \cdot [\cosh(pt_1\delta) - \omega]}$$

$$\omega = \frac{\sinh(pt_1\delta)}{pt_1\delta}$$

$$\delta = \frac{k-1}{k+1}$$

$$pt = 2pt_1 \cdot \delta \cdot \frac{1}{{}^e\log k}$$

$$pt_0 = 2pt_{01} \cdot \delta \cdot \frac{1}{{}^e\log k}$$

On the basis of these simple hypotheses the minimum for eq. 3.3 and 3.7 can be calculated analytically.

The result is illustrated in *Fig. 3.4* as regards the maximum hypothesis and in *Fig. 3.5* as regards the minimum hypothesis.

In these diagrams $k = \dfrac{c_2}{c_1}$ is the "factor of uncertainty", *i.e.* the relationship between the expected maximum and minimum rise in the need. At $k = 1$ the expressions take the form of the already derived equation

$$p\,(t_0 + t) = e^{pt} - 1 \tag{3.10}$$

Comparison between the values of pt in *Figs. 3.4* and *3.5* with those in *Figs. 2.1* and *2.2* shows that the increase in cost as a result of planning on the basis of the mean value instead of the assumed frequency functions is generally less than ½ per cent even at values of k as high as $k = 3$.

This result confirms that the model employed to describe the growing need is usable and that the assumed character of the frequency function $f(c)$ is of subordinate importance; and it at all events indicates that, in deciding the size of instalments, the frequency function can be represented by a single value corresponding to the estimated mean increase in the need c_m.

To determine the time at which an extension should be decided, it is not sufficient, however, as we shall see later, to use the estimated mean increase in the need, c_m, but the frequency function $f(c)$ must be taken into account as well.

CHAPTER 4

Estimate of Size of Plant Extensions Having Regard to the Risk of Stagnation in the Need

It has been apparent from the foregoing that the capacity of a plant extension, and so the immediate capital requirement, can be greatly reduced without the future costs increasing by more than about one per cent.

If one is doubtful about the duration and magnitude of the growth, one should obviously observe some caution in deciding on the size of the extension. Some guidance in deciding on the limits of plant capacity can be obtained merely from *Figs. 2.1* and *2.2*, from which it will be seen that, for a continuing growth in the need, the cost of the first extension can be reduced by 10—20 per cent without the future costs increasing by more than 1 per cent.

Assume by way of example that, for a continuing growth in the need, the present value of the most economic extension is 100 and the cost of the first extension is 63. For a 20 per cent reduction of the capacity of the first extension, the cost of the first extension falls from 63 to 56. Through this saving on the first extension, however, the costs under conditions of continuous growth rise to 101.5.

The costs of these two alternatives may then be as follows:

Plant expansion	1 Need stagnates	2 Need continues to grow
Alternative 1..................	63	100
” 2..................	56	101.5

6—102993

Insofar as we have an idea of the probability S_1 that the need will stagnate, and so of the probability $S_2 = 1 - S_1$ that the need will continue to grow, we can calculate which alternative is likely to result in the least cost by comparing the two expressions

$$63\,S_1 + 100\,S_2$$
$$56\,S_1 + 101.5\,S_2$$

If one has only a vague idea of these probabilities, it is a good help to determine the probabilities S_1 and S_2 which yield the same anticipated costs under the two alternatives, and which, if they conform with reality, will give no priority to either alternative, so that the choice of method is a matter of indifference.

In the example above, this is achieved for $S_1 = 17$ per cent and $S_2 = 83$ per cent, which shows that it is advisable to reduce the extension by 20 per cent, and so the immediate capital requirement by 11 per cent, as soon as the probability of stagnation is judged to be above 17 per cent.

If one has no idea whatsoever about this probability, it is generally best to choose the least extension, that is alternative 2, in the example above.

It may be of some interest to generalize the above example.

Assume that the most economic instalment for a growth c is

$$a + bct \tag{4.1}$$

and that, owing to some doubt concerning the duration of the growth, one considers reducing that instalment to

$$a + bc(1 - \eta) \cdot t \tag{4.2}$$

On the basis of these data and of the general minimum conditions (see, for example, chap. 1), we set up the matrix of the costs shown below.

Costs of different alternatives for plant expansion and growing need

		Need 1: Stagnates before full capacity utilized in first stage	Need 2: Continues to grow
Expansion	1. period of provision $(1-\eta)\,t$	$a + bc(1-\eta) \cdot t$	$\dfrac{a + bc(1-\eta)\,t}{1 - e^{-p(1-\eta)\,t}}$
	2. period of provision t	$a + bc\,t$	$\dfrac{a + bc\,t}{1 - e^{-pt}}$

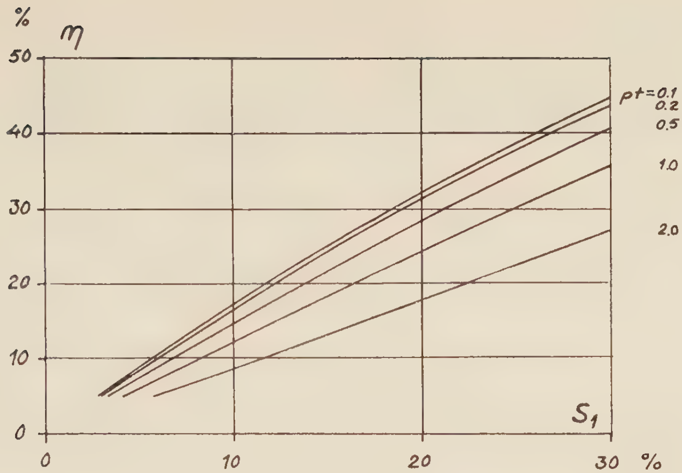

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Fig. 4.1. Reduction η, in the capacity of an instalment as function of S_1, the probability of stagnation in the growth of the need.

$$S_1 = \frac{1 - e^{-\eta pt} \cdot [1 + \eta pt]}{1 - e^{-\eta pt} - \eta pt \cdot e^{-pt}}$$

By using the minimum condition

$$pt_0 + pt = e^{pt} - 1$$

$$t_0 = \frac{a}{bc}$$

subtraction of $a + bc \cdot t$ and multiplication by $\frac{p}{bc}$ the matrix is transformed to

$$\begin{Bmatrix} -\delta_1 & 1 + \delta_2 \\ 0 & 1 \end{Bmatrix} \quad \text{where } \delta_1 = \eta pt \qquad \delta_2 = \frac{1 - e^{-\delta_1}(1 + \delta_1)}{e^{-\delta_1} - e^{-pt}} \tag{4.3}$$

This shows that, for a reduction of the extension by the factor η, one makes a saving on the first instalment of δ_1, with the result that the cost increases by δ_2 under conditions of continuously increasing need.

If one assesses the probability of stagnation as S_1, and of continued growth as S_2, then a reduction by the factor η on the first instalment can be made as soon as

$$\frac{S_1}{S_2} > \frac{\delta_2}{\delta_1} \tag{4.4}$$

Fig. 4.1 shows the factor η as function of S_1 for some different values of pt.

Clearly considerable reductions in the extensions can be made even if the probability of stagnation is judged to be comparatively small. At, for example, $S_1 \geq {}^1/_{10}$ and $pt = 0.5$, the extensions can be reduced by more than 15 per cent and considerable savings can be made in the immediate capital requirement without any great risk of adding to the future expenditure.

CHAPTER 5

Choice Between Extension in One and Extension in Two Stages

The methods discussed in the preceding chapter generally afford fully satisfactory guidance for determining the degree to which a plant should be extended in order to meet a growing need in the best possible manner.

Although these methods rest for the main part on the assumption of a non-stagnating development over a long period, they can also be used for many applications when the period studied is limited to a defined interval $0 - T$.

In such a case, insofar as the result obtained by these methods indicates that two or more extensions should be undertaken during the period, the calculations will at all events not need to be supplemented owing to the flatness of the minimum.

Only if the stage falling within a delimited period is a complete one may it sometimes be necessary to pursue the investigation further in order to establish whether the extension should be undertaken in one or two stages during the period under consideration.

In an investigation of this kind, with a plane horizon T, the costs of extensions which may be required after that time are excluded, either because the need for an individual plant component is then considered to be saturated and new extensions are very unlikely, or quite simply because one wishes to study the development within a period in which the need can be predicted with greater assurance.

In the sequel we shall distinguish between two cases, *viz.*

1. Limited number of choices of rate of expansion.
2. Unlimited number of choices of rate of expansion.

1. *Limited number of choices of rate of expansion*

The alternative courses of action can always be investigated *in pairs* and the best result obtained by elimination.

For this reason it will suffice to describe a method for the simple case that there are only two choices:

a) to extend the plant to a capacity at $t = 0$ sufficient to cover the requirement during the entire period $0 - T$ and at a cost A_0;

b) to extend the plant to a lower capacity at $t = 0$ at a cost A_1 and to add to it at a future time t at a cost A_2.

The assumption is naturally that

$$\Delta k = A_1 + A_2 - A_0 > 0 \tag{5.1}$$

Expansion in one stage will then be more advantageous than expansion in two stages as soon as

$$\frac{\Delta k}{A_2} > 1 - e^{-pt} \tag{5.2}$$

This inequality is illustrated in *Fig. 5.1*, from which can be read the least period of time within which a second instalment will be required if expansion in two stages according to alternative b) is to be profitable.

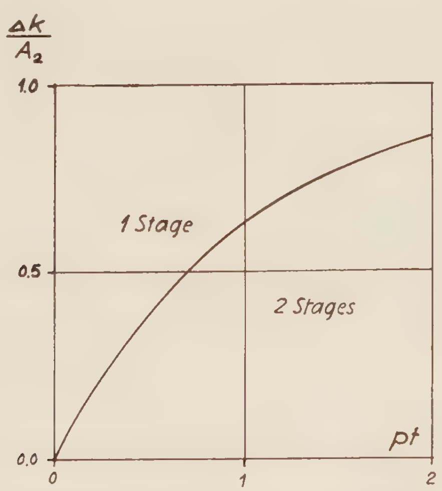

Fig. 5.1. The choice between two possible alternative courses of action

$$\frac{\Delta k}{A_2} \lesseqgtr 1 - e^{-pt}$$

2. *Unlimited number of choices of rate of expansion*

With a large or unlimited number of choices, one should attempt to summarize the costs in a relationship having, for example, the form

$$a + b \cdot n$$

where a and b are constants and n is the size of extension. For a linear increase in the need c the cost of an instalment which covers the need during the entire period planned for will then be

$$N_1 = a + bcT \tag{5.3}$$

whereas the cost for two stages will be

$$N_2 = a + bct + [a + bc\,(T - t)]\, e^{-pt} \tag{5.4}$$

where t is the economic period of provision for the first stage. N_2 will be minimum when

$$pt_0 = e^{pt} - 1 - (T - t) \cdot p \tag{5.5}$$

where

$$t_0 = \frac{a}{bc}$$

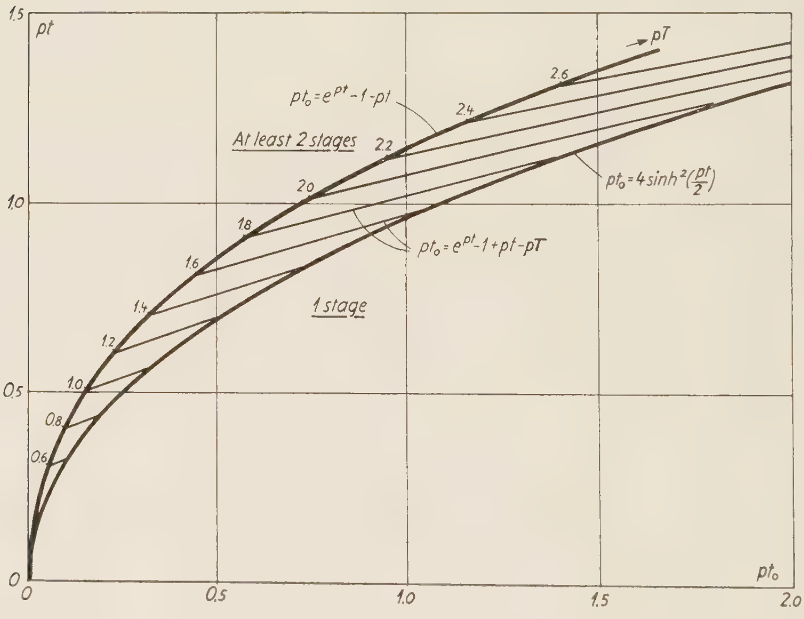

Fig. 5.2. Guidance for deciding whether a plant should be built in one or two stages, and in the latter case the relative quantities to be allotted to each.

and, comparing N_1 and N_2, we find that a single stage is preferable to two stages as soon as

$$\boxed{pt_0 > 4\sinh^2\left(\frac{pt}{2}\right)} \tag{5.6}$$

Fig. 5.2 is drawn with the aid of eq. 5.5 and 5.6 and gives guidance for deciding whether the plant should be built in one or two stages, and in the latter case the relative quantities to be allotted to each.

CHAPTER 6

Extension of Plant Serving a Growing Number of Subscribers

In this chapter we shall apply the general principles developed in chapter 1 to a concrete case, namely, to the question of when a decision should be made to extend a plant serving a number of subscribers which grows with time. The same question, but relating to a growing traffic, will be dealt with in the next chapter.

A plant is assumed to have a reserve of n_0 lines at the time of calculation $t = 0$. At a future time t the growth of the need after time $t = 0$ is assumed to be ν with a probability $P(\nu, t)$. The cost of the extension, K kronor, is assumed to be known.

To determine the time t_x at which the extension should be built, it is sufficient, as will appear from chap. 1.2, to know the intensity of the inconvenience at a given point of time.

In the present case this inconvenience is measured by the expected number of subscribers who cannot be connected at time t, *i.e.* the expected number of queuing subscribers, which is

$$\sum_{\nu = n_0+1}^{\infty} (\nu - n_0) \cdot P(\nu, t) \tag{6.1}$$

and the inconvenience function can be written

$$\Psi(n_0, t) = pGT_0 \sum_{\nu = n_0+1}^{\infty} (\nu - n_0) \cdot P(\nu, t) \tag{6.2}$$

where, as in the paper *Economic Optimum for Conductor Diameter and Number of Switches on Telephone Circuits,*

G = quality of service factor which expresses the capitalized value of one hour per day of the subscriber's time

$T_0 = \frac{24}{N_t} \cdot \int_0^{\infty} A \cdot f_t(A) \cdot dA$ = the total conversation time of the subscriber per day, hours

$f_t(A)$ = a frequency function describing the traffic during one year at time t

N_t = the number of subscribers at time t

$p = \ln(1+r)$

r = the interest factor

In this context T_0 is assumed to be constant during the period relevant to the study.

On the assumption that the inconvenience after expansion of the plant can be disregarded (cf eq. 1.11), the time t_x for the extension is determined by the equation

$$\boxed{p \cdot K = \Psi(n_0, t_x)} \tag{6.3}$$

In the sequel we shall illustrate the method by making two special assumptions concerning the character of the growing need.

1. The future growth, c subscribers per annum, assumes with equal probability any value whatsoever

$$c_1 \leq c \leq c_2$$

between a lower limit c_1 and an upper limit c_2. On this assumption the frequency function for the subscriber growth is

$$f(c) = \frac{1}{c_2 - c_1} = \frac{1}{2\,c_m} \cdot \frac{k+1}{k-1} \tag{6.4}$$

with mean growth $c_m = \frac{c_1 + c_2}{2}$

and the "factor of uncertainty" $k = \frac{c_2}{c_1}$

The time t_x for extension of the plant at a cost of K kronor is determined by the equation

$$p \cdot K = p \cdot G \cdot T_0 \int_{n_0/t_x}^{c_2} (ct_x - n_0) \cdot \frac{dc}{c_2 - c_1} \tag{6.5}$$

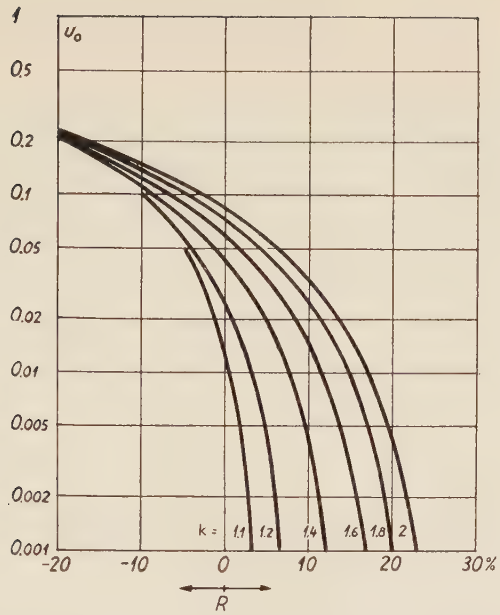

Fig. 6.1. Aid for decisions concerning the appropriate date to order an extension

$$u_0 = \frac{K}{n_0} \cdot \frac{1}{GT_0} = \frac{\left[1 - (1-R) \cdot \dfrac{2k}{k+1}\right]^2}{4(1-R)\dfrac{k-1}{k+1}}$$

$k = 1.1,\ 1.2,\ 1.4,\ 1.6,\ 1.8,\ 2$

K = cost of extension

n_0 = the reserve at the date of calculation, $t = 0$

GT_0 = the value of a subscriber's conversation time

R = the reserve which should be available at the time $t = t_x$, when the extension is ready for use

$k = \frac{c_2}{c_1}$ = the factor of uncertainty regarding the future growth of the need

c_2 = upper limit för the annual growth

c_1 = lower limit for the annual growth

t_x is obtained from the expression

$$t_x = \frac{1-R}{c_m} \cdot n_0$$

where

$$c_m = \frac{c_1 + c_2}{2}$$

If delivery time is t_d, the extension should be ordered at the point of time

$$t = t_x - t_d$$

7—102993

Introducing

$$R = \frac{n_0 - c_m t_x}{n_0} \tag{6.6}$$

where

R = the reserve which should be available at time $t = t_x$ in relation to the reserve n_0 at time $t = 0$,

we obtain after some simplifications the following expressions for determination of R and t_x:

$$u_0 = \frac{K}{n_0} \cdot \frac{1}{GT_0} = \frac{\left[1 - (1-R) \cdot \dfrac{2k}{k+1}\right]^2}{4\,(1-R)\,\dfrac{k-1}{k+1}} \tag{6.7}$$

$$t_x = \frac{1-R}{c_m} \cdot n_0 \tag{6.8}$$

which hold good under the normally fulfilled assumption that $n_0 > c_1 t_x$. The relationship in eq. 6.7 is represented in *Fig. 6.1.*

2. The probability $P(\nu, t)$ for a growth of ν subscribers during the period $0 - t$ is assumed to be describable by a Poisson distribution

$$P(\nu, t) = \frac{\lambda_t^\nu}{\nu!} e^{-\lambda_t} \tag{6.9}$$

where

λ_t = the estimated mean growth during the period $0 - t$.

The expected number of queuing subscribers at time t is then

$$\sum_{\nu = n_0+1}^{\infty} (\nu - n_0) \cdot \frac{\lambda_t^\nu}{\nu!} e^{-\lambda_t} \tag{6.10}$$

and the time t_x for extension of the plant is determined from the equation

$$u_0 = \frac{K}{n_0} \cdot \frac{1}{GT_0} = \frac{1}{n_0} \sum_{n_0+1}^{\infty} (\nu - n_0) \cdot \frac{\lambda_{t_x}^\nu}{\nu!} e^{-\lambda_{t_x}} \tag{6.11}$$

The right-hand term of this equation can easily be simplified into an expression suited for numerical calculations. This gives

$$u_0 = \frac{\lambda_{t_x}}{n_0} \sum_{n_0}^{\infty} \frac{\lambda_{t_x}^\nu}{\nu!} e^{-\lambda_{t_x}} - \sum_{n_0+1}^{\infty} \frac{\lambda_{t_x}^\nu}{\nu!} e^{-\lambda_{t_x}} \tag{6.12}$$

For large values of λt_x this expression can with good approximation be written

$$u_0 = \frac{\lambda_{t_x}}{n_0} \Phi \left(\frac{\lambda_{t_x} + 0.5 - n_0}{\sqrt{\lambda_{t_x}}} \right) - \Phi \left(\frac{\lambda_{t_x} - 0.5 - n_0}{\sqrt{\lambda_{t_x}}} \right) \tag{6.13}$$

with

$$\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \cdot dx$$

These examples of the method of determining the time for extension will hereafter be illustrated numerically.

Assume that a telephone exchange with a capacity of 10,000 lines at time $t = 0$ has 6,000 lines installed. The net requirement is estimated to grow by $c_m = 1{,}000$ lines per annum with a factor of uncertainty $k = \frac{c_2}{c_1} = 2$. The calculation of the most appropriate quantity of equipment to be added is assumed to have resulted in a cost of $K = 3.2$ million kronor. GT_0 is evaluated at 80,000 kr.

Under these assumptions $n_0 = 4{,}000$ and

$$u_0 = \frac{3.2 \cdot 10^6}{4 \cdot 10^3 \cdot 8 \cdot 10^4} = 0.01$$

Reading from the diagram in *Fig. 6.1*,

$$R = 17\,\%$$

corresponding to

$$t_x = \frac{1 - 0.17}{1{,}000} \cdot 4{,}000 = 3.3 \text{ years}$$

If the time of delivery is 2.5 years, the decision to instal this quantity of equipment can be slightly postponed.

After some time the reserve has fallen to $n_0 = 3{,}200$ and, in the same way as above, we get

$$u_0 = 0{,}0125,\ R = 16\,\%,\ t_x = 2.7 \text{ years}$$

signifying that the decision to instal the equipment should be made very soon.

The expected number of queuing subscribers at $t_x = 2.7$ years is

$$n_0 u_0 = 3{,}200 \cdot 0.0125 = 40,$$

and the calculated reserve over and above the estimated mean requirement is 500 lines, which is accordingly the "cushion" to be allowed in planning.

It may be of interest to make a corresponding calculation on the assumption of the subscriber growth being describable by a Poisson distribution.

Supposing $\lambda_{t_x} = \underline{3{,}230}$

one obtains $\dfrac{\lambda_{t_x}}{n_0} = 1.009375, \qquad \sqrt{\lambda_{t_x}} = 56.833$

$$\Phi\left(\frac{30.5}{56.833}\right) = 0.7043, \qquad \Phi\left(\frac{29.5}{56.833}\right) = 0.6982$$

$$u_0 = 1.009375 \cdot 0.7043 - 0.6982 = 0.0117$$

Supposing $\lambda_{t_x} = \underline{3{,}235}$ one obtains $u_0 = 0.0139$

This shows that for a mean increase of 1,000 lines per annum the plant should be extended in about 3.2 years, *i.e.* 5 months later than had been previously calculated under other assumptions concerning the character of the subscriber growth. At this time the anticipated number of subscribers exceeds the capacity of the plant by about 1 per cent.

The difference is due primarily to the fact that the variances of the two distributions describing the subscriber growth differ very greatly and in fact are in the ratio of about 20:1.

If the decision concerning the time at which the plant should be extended is based on the hypothesis that the subscriber growth is described by a Poisson distribution, this means that one is fairly sure of being able to assess the mean growth. But this can only rarely occur. In the event of doubt concerning the magnitude of the mean growth, one might assume an ensemble of Poisson distributions and allot to each a given probability. But this is likely to complicate the calculations in a probably quite unnecessary manner. The simplest method, and one that is fully adequate for most practical applications, is to check the uncertainty of a prognosis by extrapolating historical data. In this way one narrows down the future development to a sector in which the need is likely to lie. The width of this sector determines the factor of uncertainty which can be used for the calculations in accordance with hypothesis 1 above. In our example this factor of uncertainty was put very high. For periods as short as 2—3 years one should usually be able to employ a considerably lower value.

Hitherto we have assumed that the capacity and cost of the planned extension can be determined without regard for its time. As we saw in chapter 1, the size and time for installation of the plant must in principle be determined simultaneously. In reality, however, it is fully adequate first to determine the most suitable size of extension based on the estimated mean growth and on the cost of extensions of different capacity, without any loss of accuracy being thereby incurred.

The truth of this supposition can be checked in a simple way by studying the expression

$$\frac{1}{1-e^{-pT}}\left[a+b\cdot x+p\,GT_0\int_{x/c_0}^{T}(c_0t-x)\cdot e^{-pt}\cdot dt\right] \qquad (6.14)$$

which represents the present value of the costs of a series of extensions of capacity x at regular intervals of time T, including the present value of the inconvenience to subscribers for an increase in the need of c_0 lines per annum. The conditions necessary for this expression to be minimum are

$$\frac{b}{GT_0}=e^{-p\frac{x}{c_0}}-e^{-pT} \qquad (6.15)$$

$$a+bx+\frac{bc_0}{P}=GT_0\,(c_0T-x) \qquad (6.16)$$

Elimination of x gives

$$p\,(t_0+T)=\frac{1+\varepsilon}{\varepsilon}\ln\left[1+\varepsilon\cdot e^{pT}\right]-1 \qquad (6.17)$$

with

$$\varepsilon=\frac{b}{GT_0}$$

and

$$t_0=\frac{a}{bc_0}$$

For $G\rightarrow\infty$ this expression takes the form already derived in chapter 1:

$$p\,(t_0+T)=e^{pT}-1 \qquad (6.18)$$

The difference between the economic period of provision T calculated as in eq. 6.17 and that calculated as in eq. 6.18 is negligible so long as ε, as is usually the case, is less than 0.05.

CHAPTER 7

Extension of Plant to Provide for Growing Traffic

The time at which a decision should be made to extend a plant, having regard to the inconveniences suffered by subscribers under conditions of growing traffic, must be determined on the same principles as those described in the preceding chapter for a growing number of subscribers.

The inconvenience function in the present case, however, is clearly

$$\Psi(n, t) = pGT \int_0^\infty A \cdot E(n, A) \cdot f_t(A) \cdot dA \tag{7.1}$$

where

$f_t(A)$ = the frequency function for the traffic, A, applying for one year at time t. This frequency function is in principle formed by proceeding from an ensemble of all conceivable frequency functions at that time and allotting to each a certain probability $P_t[f(A)]$.

T = the time expressed in hours per day described by the frequency functions $f(A)$

$E(n, A)$ = the congestion at a traffic A with n switches installed

p = $\ln(1+r)$

r = the interest factor

Say that a plant containing n_0 switches is in service and is to be extended by a group of n_1 switches at a cost K at time t_x. According to eq. 1.11 this time can be immediately determined from

$$\Psi(n_0, t_x) - \Psi(n_0 + n_1, t_x) = p \cdot K \tag{7.2}$$

If $n_1 = 1$ and $K = b$, we get the relation known from Part I, *Economic Optimum for Conductor Diameter and Number of Switches on Telephone Circuits:*

$$\varepsilon = \frac{b}{G} \approx T \int_0^\infty A\Delta E \cdot f_t(A) \cdot dA \tag{7.3}$$

where

$$A\Delta E = A\,[E(n_0, A) - E(n_0 + 1, A)]$$

If n_1 is large, the traffic losses after expansion become negligible, and t_x can with very good approximation be determined from

$$\boxed{\Psi(n_0, t_x) = p \cdot K} \tag{7.4}$$

Say that one wishes to determine the time for extension of an already operating plant having a capacity of $n_0 = 25$ circuits. Ten circuits are to be added at a cost of 65,000 kronor. G is estimated at 100,000 kronor.

The weighted frequency function for the traffic during a period of $T = 2.2$ hours per day has been estimated at the values given in *Table 1* for times $t = 2.75$, $t = 3.00$ and $t = 3.25$ years.

Table 1

A	$t = 2.75$ $P_t(A)$	$t = 3.00$ $P_t(A)$	$t = 3.25$ $P_t(A)$
10	—	—	
11	0.02	0.01	—
12	0.06	0.05	0.01
13	0.11	0.09	0.03
14	0.16	0.13	0.07
15	0.17	0.16	0.11
16	0.16	0.16	0.15
17	0.13	0.14	0.16
18	0.08	0.10	0.15
19	0.05	0.07	0.12
20	0.03	0.04	0.09
21	0.02	0.03	0.06
22	0.01	0.02	0.03
23	—	—	0.02
24	—	—	—
25	—	—	—

Hence G can be calculated for different values of t as follows.

Table 2

t	$T\Sigma[AE \cdot P(A)]$	$G = \dfrac{65.000}{T\Sigma[AE \cdot P(A)]}$
2.75	0.48	136,000 kr
3.00	0.64	102,000 kr
3.25	0.82	80,000 kr

The plant should therefore be extended after a period of about 3 years. If the delivery period is 2 years, no immediate decision need be made, but the calculation can be revised later on when the trend can be better foreseen.

This method of determining the time for a plant extension can be used also for assessing the relative merits of alternative schemes, for example whether an extension should be installed in one or more stages. The times for the various instalments are determined in exactly the same way as above, the present values of the total costs of the various alternatives being calculated and compared so as to arrive at the most suitable scheme.

For certain kinds of plant, for example a trunk cable, the choices are so numerous that it is worth making a direct approximate calculation of the size of the instalment. As appeared from chap. 1, this can easily be done if, as is usually the case, the cost can be represented by a linear function,

$$k = a + b \cdot n$$

where a and b are constants and n is the need.

In the previous example (see *Table 1* and *2*) the mean values of the traffic at $t = 2.75$, 3.00, and 3.25, are $A_m = 15.5$, 16 and 16.5, corresponding to an annual mean growth of 2 traffic units. We assume that this traffic growth continues and wish to estimate the mean growth, c, in number of circuits per annum. This can be done by assuming that the frequency function $f_t(A)$ can be represented by its mean value and that the corresponding congestion E' immediately before expansion of the plant lies within certain very wide limits. Let us assume in the present example that $0.001 < E' < 0.1$. The number of switches calculated with the aid of Erlang's formula for lost call systems is then:

Table 3

t	A_m	n	
		$E' = 0.1$	$E' = 0.001$
0	10	13	21
5	20	23	35
10	30	32	47

and the growth in the number of switches per annum under this assumption is $2 < c < 2.8$.

Assuming $c = 2$, we get for $a = 15{,}000$ kr, $b = 5{,}000$ kr, $p = 0.08$,

$$pt_0 = \frac{a}{b \cdot c} \cdot p = 0.12$$

whence, from *Fig. 2.2*, chap. 2,

$$pt = 0.45 \qquad t = 5.63 \text{ years}$$

and the size of the instalment is $2 \cdot 5.63 = 11.3$ pairs.

For $c = 2.8$, the size of the instalment will be 13.4 pairs.

From this calculation we see that the size of the instalment, n_1 lies within the limits

$$11 < n_1 < 14$$

provided that the traffic losses at the times of the instalments, based on the frequency function $f_t(A)$, are within the limits obtained by proceeding from the mean value of the traffic and the assumed congestions 0.1 and 0.001.

For $11_1 < n_1 < 14$ the plant must be replaced at a time t, so that

$$3 < t < 3.25$$

and the traffic losses at that time are accordingly

$$0.64 < T \sum AE \cdot P(A) < 0.82$$

But at time t, consequent on the given assumptions,

$$0.02 < T \cdot A_m \cdot E' < 3.6$$

which confirms the reasonableness of the assumptions.

If required, these limits can be narrowed and the accuracy in estimating the size of the extension can be slightly improved.

This method should be sufficiently accurate for practical requirements for determining the time and size of a future extension to cater for a growing traffic.

If greater accuracy is required, use must be made of the general equations 1.5 in chapter 1.

For the present purpose they may be summarized in the following formulæ using eq. 7.4:

$$\left.\begin{array}{l} GT \cdot AE(M_{\nu-1}, t_\nu) = a + bn_\nu \\ pGT \displaystyle\int_{t_\nu}^{t_\nu+1} A \Delta E(M_\nu, t) \cdot e^{-pt} \cdot dt = b\,(e^{-pt_\nu} - e^{-pt_\nu+1}) \\ M_\nu = M_{\nu-1} + n_\nu, \quad M_0 = n_0, \\ (\nu = 1,2\ldots) \end{array}\right\} \qquad (7.5)$$

For the sake of simplicity the frequency function $f(A)$ has been replaced in these formulæ by a single value applying during T hours per day, which can obviously be done without loss of general validity.

Fig. 7.1 illustrates the relations between $n_1, n_2 \ldots t_1, t_2 \ldots$ under the following assumptions:

$A = 10 + 2t$ erlang

$a = 15{,}000$ kr $\qquad b = 5{,}000$ kr

$GT = 220{,}000$ kr

$p = 0.08$

E = the congestion in a lost call system with full availability.

If in this figure we consider initially the relation between n_1 and t_1, representing the size and time of the immediately pending extension, which are the only factors of interest at the time of calculation, we see that a slight change in t_1 causes a considerable variation in n_1. For an increase of t_1 from 3.6 to 4.2 years, n_1 increases from 10 to 20 pairs. This suggests that it is more important that the extension should be made at the right time than that it should be made of the right size.

It was shown in chapter 2 that the size of instalments could vary within fairly wide limits without the long-term costs changing by more than one or so per cent. This was true provided that the determination of the size of instalments was based solely on the growth of the need counted in number of switches or lines per annum.

A study with the aid of *Fig. 7.1* shows that the cost minimum will be still flatter if the economic value of the inconvenience to subscribers is taken into account as well as the cost structure when determining the size of instalments. This is quite natural when it is considered that the cost of extensions and the economic value of the inconvenience can to some extent be substituted for one another.

As stated in chapter 1, there is a choice of two methods for estimation of n_1 and t_1.

1. In place of the cost of an extension, $a + b \cdot n_\nu$, use can be made of the estimated value of future extensions reduced to the time t_ν. In the numerical example above, if the growth is estimated as $c = 2$, this gives

$$N = \frac{bc}{p} \cdot e^{pt_x} = \frac{5{,}000 \cdot 2}{0.08} \cdot e^{0.45} = 196{,}000$$

Assuming further that only two stages are included, trial and error calculations give

$$n_1 = 14,\ t_1 = 3.85,\ n_2 = 11,\ t_2 = 9.2,\ t_3 = 15.3$$

2. Costs arising after a time t_h, the plane horizon, are disregarded. The following values are obtainable from *Fig. 7.1* for different values of t_h:

t_h	extension in one stage	extension in two stages	
	n_1	n_1	n_2
10	16	—	—
12	20	13	9
14	—	15	11
16	—	17	14

For higher values of t_h it pays to build in three or more stages.

One can get a good idea of the flatness of the cost minimum by studying the expression

$$N_0 = \frac{1}{1 - e^{-pT}} \left[a + b \cdot n + pGT \int_0^T A \cdot E \cdot e^{-pt} \cdot dt \right] \tag{7.6}$$

which indicates the present value of the costs of instalments and the value of the inconveniences provided that the instalments are of equal size and that the inconveniences are repeated at periods T.

For a given value of n, this expression becomes minimum as soon as

$$N_0 = GT \cdot AE \tag{7.7}$$

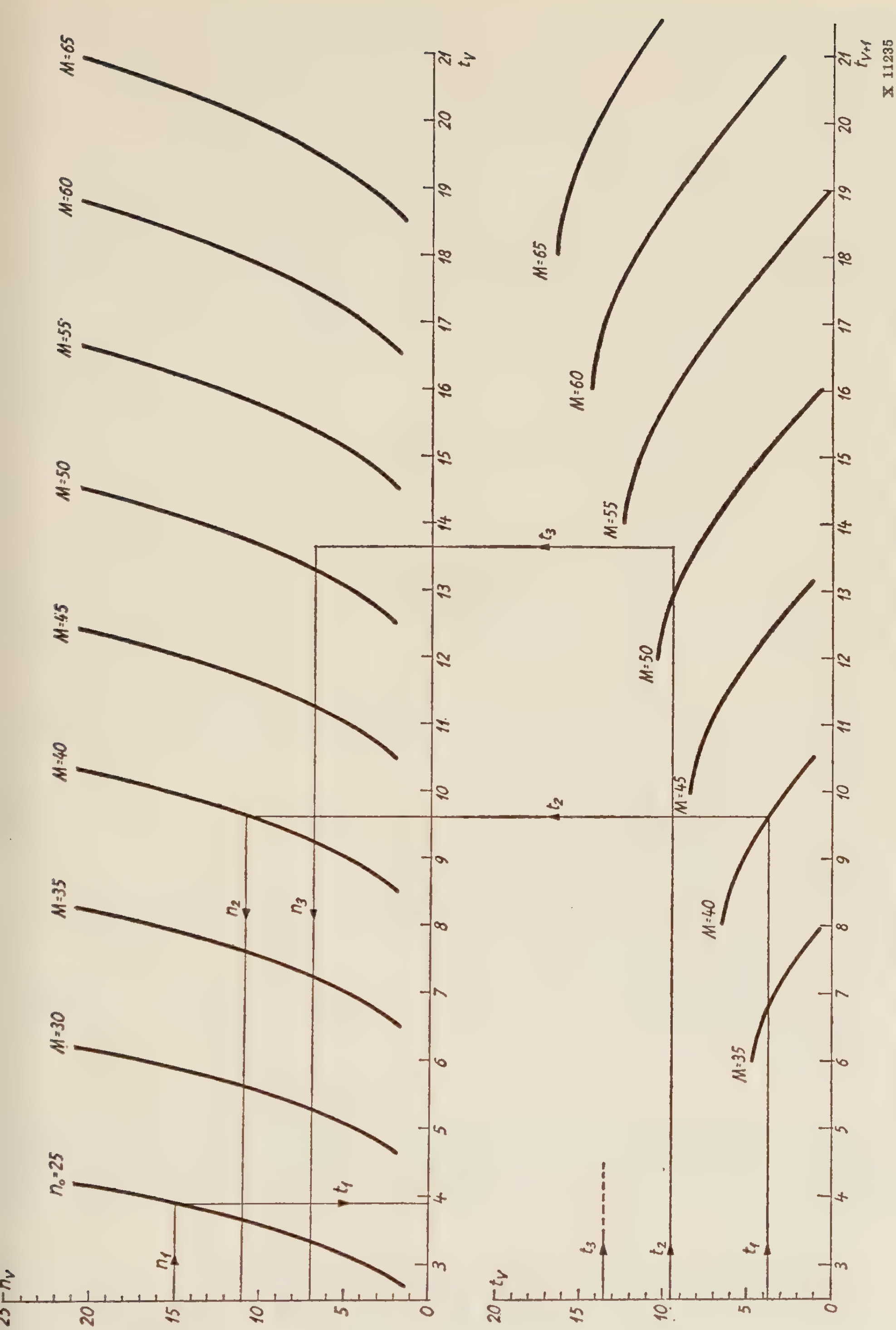

Fig. 7.1. The relations between n_1, n_2 . . . . t_1, t_2 . . . under certain assumptions mentioned in the text.

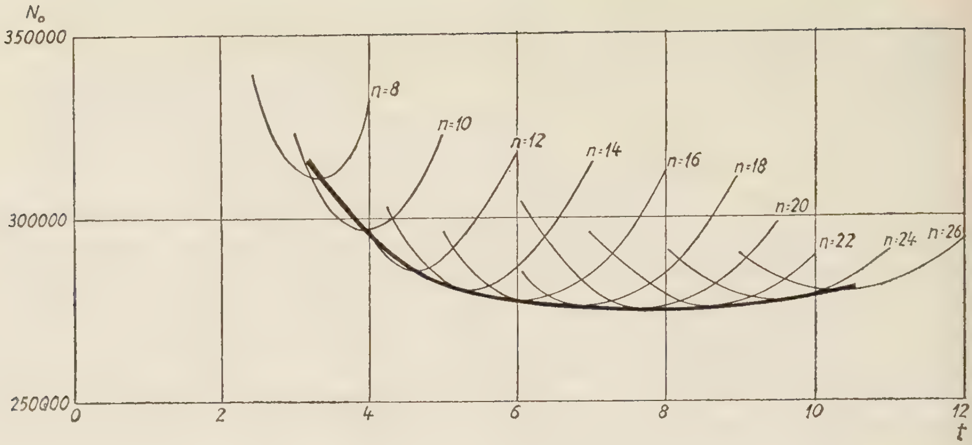

Fig. 7.2. The character of the minimum of the function

$$N_0 = f(n, T) = \frac{1}{1 - e^{-pT}} \left[a + bn + pGT \int_0^T AE \cdot e^{-pt} \cdot dt \right]$$

Fig. 7.2 shows N_0 as function of n and T with the numerical values used in the foregoing.

The studies made in this chapter suggest that, in determining the time and size of an extension which is to provide for a growing traffic requirement, it suffices first to determine its size and thereafter the time for its installation.

The size of the extension is determined by

replacing the frequency function $f_t(A)$ by a single value $A_* = A_*(t)$,

assuming on trial that the congestion immediately before installation is within given limits,

estimating therefrom and from the increase in the "equivalent traffic" A_* the growth of the need in number of switches or lines,

and hence determining the size of the instalments by means of the methods outlined in chapters 1—5.

After the size of the extension has been estimated in this way and finally established in respect of standard designs of equipment, availability of capital, etc., the time at which the decision to install the extension should be made must be determined in the manner described in this chapter.

If, as has been done consistently in this paper, the subscribers' demands for accessibility are considered as an economic question and their time is ascribed a value which exceeds

the cost of the telephone connections, manifestly a greater mistake will generally be made from the social-economic point of view if one decides on an extension at the wrong time than if one makes a somewhat wrong decision about its size.

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A Theory of the Eddy Current Equivalent Winding and its Application to the Closing of Non-Delayed Telephone Relays

A Study of Telephone Relays (4)

BY

STIG EKELÖF*

		Page
Chapter 1	Introduction and summary	104
Chapter 2	General theory of the eddy current equivalent winding	105
Chapter 3	Parameters of the eddy current equivalent winding of a relay on closing .	114
Chapter 4	The closing of relays without magnetic leakage and with eddy currents in the core only	124
Chapter 5	The closing of relays with magnetic leakage and with eddy currents in core and yoke	132
Bibliography	. . . .	134
Principal symbols and formulae	. . . .	135

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In this paper the influence of the eddy currents on a telephone relay of the electromagnetic type on closing is investigated. The same relay as in Parts 1, 2, and 3 of the study is considered, i.e., an "L-armature" relay with a round core and a rectangular yoke, where the magnetic leakage is taken care of by the simplified field picture introduced in Part 1. The relay is assumed to have eddy currents in core and yoke. The armature is kept fixed and the magnetic reluctances are supposed to be constant.

The concept of an equivalent winding, representing the eddy current paths in a magnetic circuit, is subjected to a general theoretical treatment. It is shown that an equivalent winding should give useful results on closing but cannot be employed on breaking.

For the relay under consideration the equivalent winding parameters on closing are deduced. Numerical results are reported. The importance of the magnetic skin effect is pointed out.

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CHAPTER 1

Introduction and summary

In the present Part 4 of "A Study of Telephone Relays" the same type of relay is considered as in the earlier Parts 1, 2, and 3[2,3,4], *i.e.*, a relay of the L-armature type with a round core and a rectangular yoke. As before, the armature is kept fixed, the magnetic reluctances are assumed to be constant, and the magnetic leakage is represented by a leakage flux, running perpendicularly between core and yoke.

The present paper and Part 3 are closely connected inasmuch as both papers discuss the same aspect of relay theory, *viz.*, the influence of the eddy currents arising in the metallic parts of the magnetic circuit. As in Part 3 we assume eddy currents in both core and yoke. For some general points of view reference is made to Part 3, Chapter 1.

Employing the concept of operational reluctances, we could in Part 3 derive without difficulty general operational expressions for the magnetic core flux on closing or breaking, and for the magnetizing primary current on closing. On *breaking*—the problem studied in detail in Part 3—it was then possible to deduce from the operator of the magnetic flux (only the armature end flux was considered) two different expressions for the corresponding time function: Heaviside's expansion theorem gave us an exponential function series, suitable for large values of the time t; a series expansion of the operator in powers of $p^{-1/4}$ gave us a series in powers of $t^{1/4}$, suitable for small t. These two solutions were used for studying quantitatively an actual relay. The results yielded by the study were in excellent agreement with actual oscillograph records of the flux decay.

In this paper we turn to a detailed study of the problem of *closing*, trying to obtain suitable expressions for the primary current and the magnetic flux. Unfortunately, in this case the method of Part 3 seems to be excluded on account of numerical complications (as to the expansion theorem solution see the remark following eq. (119) in Part 3; as to the power series solution, given by eq. (124), this solution can only be used for extremely small values of t, and further terms of the series are difficult to obtain, as they require a better approximation of the quantity $\bar{\eta}^2$ than the one given by (72)).

A new approach is necessary. In the following it will be shown that on closing sufficiently accurate solutions can be obtained by *the method of replacing the eddy current paths by an equivalent winding.*

The idea of an equivalent winding was mentioned already in Part 3, Chapter 1. It was pointed out that some earlier writers[7,8,9] have made use of this idea, interpreting more or less explicitly the equivalent winding as a physical wire winding, wound around the relay core (*cf.* also a paper by H. Hess and E. Weber[6]).

The results of Part 3 strongly supported the assertion that the breaking problem cannot be studied by the aid of a single equivalent winding. This procedure leads to an exponential decrease of the flux-time curve, an outcome which is very far from the actual course of events. Other and more suitable methods could, however, be found.

For the closing problem, on the other hand, where—as explained above—these methods break down, the concept of an equivalent winding is useful. One of the aims of the present paper is to develop a satisfactory general theory of the equivalent winding.

Chapter 2 deals with the basic principles of the equivalent winding, and gives general expressions for its parameter values, as depending upon the eddy current distribution. In *Chapter 3* these expressions are used for an approximate calculation of the equivalent winding parameters of a relay on closing.

In *Chapter 4* the results are tested on a case which can also be exactly solved, *viz.*, a relay without magnetic leakage and with eddy currents in the core only. The comparison with the exact solution turns out satisfactorily. In *Chapter 5* the equivalent winding is then used for a quantitative study of the closing of a relay *with* magnetic leakage and with eddy currents in core *and* yoke. In order to facilitate the computations, a list is finally given of the *Principal symbols and formulæ.*

CHAPTER 2

General theory of the eddy current equivalent winding

This chapter gives a general theory of the equivalent winding of the eddy currents in a magnetic circuit. The theory is presented as an application to our telephone relay. It is quite general, however, and can in principle be applied to any magnetic circuit.

We consider a relay according to *Fig. 1* on the next page (*cf.* Part 1, Fig. 4; Part 2, Fig. 9; Part 3, Fig. 2) with a cylindrical core and a rectangular yoke. The core and the yoke are assumed to be the only electrically conducting parts of the magnetic circuit. A primary magnetizing wire winding with N_1 turns is evenly distributed over the whole length of the core. The relay has a magnetic leakage flux $\varphi_\lambda(x, t)$ per unit length, running perpendicularly between core and yoke. Hence, the magnitude of the main flux $\Phi(x, t)$ passing through core and yoke varies with the distance x from the heel piece end.

If the current i_1 in the wire winding undergoes variations, we obtain through electromagnetic induction secondary or eddy currents in the electrically conducting parts of the magnetic circuit. According to Lenz' law the eddy currents give rise to opposing magnetic fluxes, which tend to slow down the rate of change of the total flux through a core section and to speed up the rate of change of the primary current.

We shall show the following. If the geometric pattern of the eddy current distribution remains constant in time, it is possible to replace the eddy currents as to their influence on the

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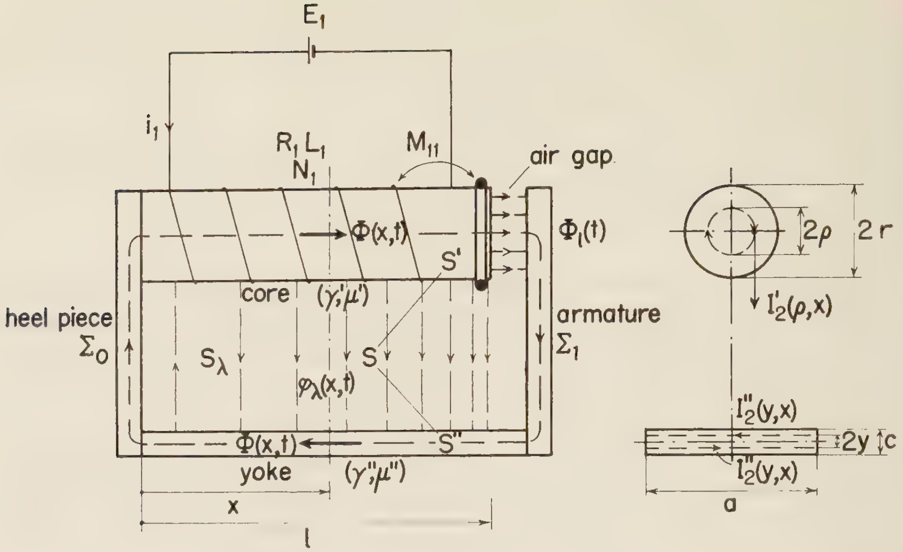

Fig. 1. Idealized relay with magnetic leakage.

primary current and on the magnetic fluxes by an equivalent winding with constant parameter values (resistance R_2, inductance L_2, mutual inductances M and M_{21}), carrying a current i_2 = the entire circulating secondary current. General expressions for the parameter values will be given.

Hence, under the condition mentioned, we can for computational purposes replace a relay with eddy currents by a relay without eddy currents but with a delaying wire winding. Thus we can employ the theory of such a relay, given in Part 2. It should be stressed that the equivalent winding is a fictitious winding, which we can use formally in our calculations even if we may not be able to realize it physically.

Units and symbols

As in the earlier parts of this study the rationalized MKS-system of units is used throughout.

In this part we denote

quantities pertaining to the core by the superscript $'$.
" " " " yoke " " " $''$.
" " " " primary (magnetizing) winding by the subscript 1.
" " " " eddy current system and the secondary (equivalent) winding by the subscript 2.

As far as possible the symbols of Parts 1 to 3 are employed. See Fig. 1 and the list of symbols at the end of the paper.

Definitions of the secondary current and of the parameters of the equivalent winding

We assume that the eddy currents in core and yoke run in planes perpendicular to the direction of x, and that their paths are symmetrical with respect to the core axis and to the symmetry planes of the yoke. This means (see Fig. 1):

the eddy current paths in the core are concentric circles; the current density I_2' is a function of the radius ϱ of the path and of its distance x from the heel piece end,

$$I_2' = I_2'(\varrho, x) \tag{1 a}$$

the eddy current paths in the yoke are parallel straight lines, with a direction parallel to the yoke surfaces; the current density I''_2 is a function of the distance y from the centre plane and of the distance x,

$$I_2'' = I_2''(y, x) \tag{1 b}$$

Of course, I_2' and I_2'' are in general also functions of the time t.

Hence, we have a circulating current in the core,

$$i_2' = \int_{x=0}^{l} \int_{\varrho=0}^{r} I_2'(\varrho, x)\, d\varrho\, dx \tag{2 a}$$

and in the yoke

$$i_2'' = \int_{x=0}^{l} \int_{y=0}^{c/2} I_2''(y, x)\, dy\, dx \tag{2 b}$$

We now define the current in the equivalent winding, the *secondary current* i_2, by the condition that it should be equal to the entire circulating current,

$$i_2 = i_2' + i_2'' \tag{3}$$

We define the resistance of the equivalent winding, the *secondary resistance* R_2, by the condition that the heat developed when the current i_2 passes through R_2 be equal to the entire eddy current heat,

$$R_2 i_2^2 = \int_{\text{core}} \frac{I_2'^2}{\gamma'}\, d\tau' + \int_{\text{yoke}} \frac{I_2''^2}{\gamma''}\, d\tau'' \tag{4}$$

We define the inductance of the equivalent winding, the *secondary inductance* L_2, by the condition that the magnetic energy arising when the current i_2 passes through L_2 be equal to the entire magnetic energy of the eddy currents. Hence, introducing the magnetic vector potentials $\mathbf{A}_2'$ and $\mathbf{A}_2''$ of the eddy current system, we shall have[1]

$$\frac{1}{2} L_2 i_2^2 = \int_{\text{core}} \frac{1}{2}\, \mathbf{I}_2' \cdot \mathbf{A}_2'\, d\tau' + \int_{\text{yoke}} \frac{1}{2}\, \mathbf{I}_2'' \cdot \mathbf{A}_2''\, d\tau'' \tag{5}$$

We finally define the *mutual inductance* M between the primary wire winding and the equivalent winding by the condition that the mutual magnetic energy of the primary current i_1 through the wire winding and the secondary current i_2 through the equivalent winding be equal to the mutual magnetic energy of the primary current and the eddy current system.

Now, as is well known, the mutual energy can be expressed in two different ways, leading to two formally different expressions for M.

If we use the vector potentials $\mathbf{A}_1'$, $\mathbf{A}_1''$ of the primary current magnetic field and the eddy current vectors $\mathbf{I}_2'$, $\mathbf{I}_2''$, we have

$$M i_1 i_2 = \int_{\text{core}} \mathbf{I}_2' \cdot \mathbf{A}_1' \, d\tau' + \int_{\text{yoke}} \mathbf{I}_2'' \cdot \mathbf{A}_1'' \, d\tau'' \tag{6 a}$$

Here the integrals should be taken over the regions which carry eddy currents, *i.e.*, over the volumes of the core and the yoke.

If, instead, we use the vector potential $\mathbf{A}_2$ of the eddy current magnetic field and the vector $\mathbf{I}_1$ of the primary current, we have

$$M i_1 i_2 = \int_{\substack{\text{primary}\\ \text{winding}}} \mathbf{I}_1 \cdot \mathbf{A}_2 \, d\tau \tag{6 b}$$

Here the integral should be taken over the region which carries the primary current, *i.e.*, over the copper volume of the primary wire winding.

Our use of the vector potentials in expressing the magnetic energies brings great advantages to our treatment inasmuch as we then only have to integrate over the current-carrying regions, while if we use expressions of the type $\int 1/2\, \mathbf{B} \cdot \mathbf{H} \, d\tau$, the integration must be extended to the entire volume of the magnetic field.

Integration of (6 b)

It is easy to solve formally the integral in (6b). It yields the expression for M given in (9), which will prove to be of great importance to us.

In carrying out the integration, we neglect the pitch of the winding, thus assuming that the current paths are circles, running perpendicularly to the core axis. We also assume that the primary winding has a very small thickness ε (see *Fig. 2*). This last assumption simplifies the calculations but is otherwise not essential.

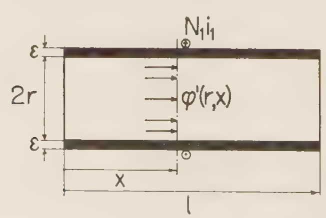

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Fig. 2. To the integration of (6 b).

Introducing in (6b)

$$I_1 = \frac{N_1 i_1}{\varepsilon l}; \quad d\tau = 2\pi r\varepsilon\, dx$$

we obtain

$$M i_2 = \int_{x=0}^{l} 2\pi r A_{2\alpha} \frac{N_1}{l}\, dx \tag{7a}$$

where $A_{2\alpha}$ denotes the component of $\mathbf{A}_2$ in the direction of $\mathbf{I}_1$, that is, in a direction tangential to the circumference of the core section.

Now, an integral $\oint A_s ds$ being equal to the magnetic flux linked with the closed path of integration, we have

$$2\pi r A_{2\alpha} = \varphi'(r, x) \tag{8}$$

where $\varphi'(r, x)$ = the total eddy current core flux at the distance x from the heel piece end. Thus, from (7a):

$$M i_2 = \int_{x=0}^{l} \frac{N_1\, dx}{l} \varphi'(r, x) \tag{7b}$$

Evidently, the last integral is equal to the total eddy current magnetic flux Φ_{21} linked with the primary winding. Hence,

$$M i_2 = \Phi_{21} \tag{9}$$

Thus, this formula, well-known from the theory of filiform circuits, is valid for our equivalent winding in the sense that the flux Mi_2 which the current in the equivalent winding sends through the primary winding, is equal to the eddy current flux Φ_{21}.

Validity of the equivalent winding

Evidently, the values of the quantities R_2, L_2, and M, which follow from (4), (5), and (6a) or (6b), are determined by the eddy current distribution at the moment in question. If our equivalent winding shall be practically useful, these values must be constant, independent of time. It is easy to see that, in accordance with our earlier prediction, *R_2, L_2, and M are constant if the geometric distribution of the eddy currents is independent of time, i.e.*, if we can write

$$I_2'(\varrho, x) = f(\varrho, x)\,\psi(t); \quad I_2''(y, x) = g(y, x)\,\psi(t)$$

with the same function $\psi(t)$ in both expressions.

If R_2, L_2, and M are constant, we can prove by energy considerations in conjunction with the relation (9) that the equivalent winding gives us the correct values of the currents i_1 and i_2.

We start with the power relation

$$E_1 i_1 = (R_1 i_1^2 + R_2 i_2^2) + \frac{dW}{dt} \tag{10}$$

expressing that the battery power $E_1 i_1$ is used up partly as Joulean heat $R_1 i_1^2 + R_2 i_2^2$, partly for increasing the magnetic energy

$$W = \frac{1}{2} L_1 i_1^2 + M i_1 i_2 + \frac{1}{2} L_2 i_2^2 \tag{11}$$

Employing (11), we can write (10) as

$$i_1 \left(R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} - E_1 \right) + i_2 \left(R_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \right) = 0 \tag{12}$$

Now, for the primary circuit we have the relation

$$R_1 i_1 + L_1 \frac{di_1}{dt} = E_1 + \left(- \frac{d\Phi_{21}}{dt} \right) \tag{13}$$

or, making use of (9):

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} - E_1 = 0 \tag{14}$$

The expression within the first pair of brackets in (12) being thus = 0, it follows that the expression within the second pair of brackets must also be = 0, *i.e.*,

$$R_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} = 0 \tag{15}$$

Evidently, (14) and (15) are the equations governing the system formed by the primary circuit and the short-circuited equivalent winding. Hence, *this coupled system yields correct values of the primary current i_1 and the total circulating eddy current i_2.*

The armature end flux

We are above all interested in the total armature end flux Φ_l. Following the procedure introduced in Part 2, Chapter 3, we express Φ_l by the aid of two mutual inductances, *viz.*,

M_{11} = the mutual inductance between the primary winding and a fictitious winding of one turn at the armature end,

M_{21} = the mutual inductance between the equivalent winding and the fictitious armature end winding.

Φ_l is produced partly by the primary current i_1, giving the contribution $M_{11}i_1$, and partly by the eddy currents. Making use of the notation $\varphi'(r,x)$ of eq. (8), we can express the contribution from the eddy currents as $\varphi'(r, l)$. We write this

$$\varphi'(r, l) = M_{21}i_2 \tag{16}$$

and take (16) as the definition of M_{21}. It is evident that M_{21} also is constant if the geometric distribution of the eddy currents is independent of time.

We thus obtain for the armature end flux

$$\Phi_l = M_{11}i_1 + M_{21}i_2 \tag{17}$$

i.e., (*cf.* Part 2, eq. (34)) the same formula as for a relay with a delaying wire winding.

A physical picture of the equivalent winding

Although it is hardly possible to materialize the equivalent winding, we can without difficulty form a physical picture of it. For simplicity we assume eddy currents in the core only. The general eddy current path then consists of a circle with the radius ϱ around the core axis.

Fig. 3 shows two perpendicular sections of the core, one through the axis and one at right angles to it.

We make use of the following symbols, dropping for the moment the superscript $'$ and the index 2.

V = core volume (volume element $d\tau$).

S = surface, on one side of core axis, of a longitudinal core section (hatched in Fig. 3; surface element da).

I = current density.

$U = 2\pi\varrho E$ = voltage around an eddy current path ($E = I/\gamma$ = electric field strength).

$\varphi = 2\pi\varrho A$ = total eddy current flux, linked with an eddy current path (A = vector potential of eddy current magnetic field).

$\Phi = 2\pi\varrho A_1$ = primary current flux, linked with an eddy current path (A_1 = vector potential of primary current magnetic field).

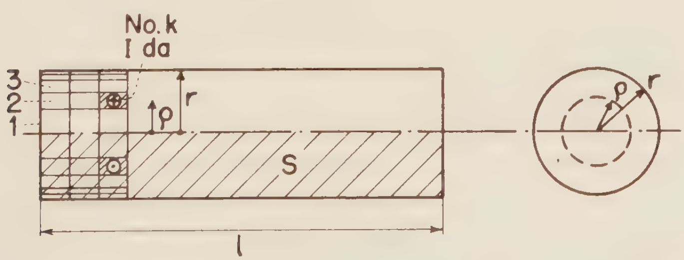

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Fig. 3. The physical realization of the equivalent winding.

According to the definition of i_2 we have

$$i_2 = \int_S I\,da \tag{18}$$

Further, eq. (4), defining R_2, gives us

$$R_2 i_2^2 = \int_V \frac{I^2}{\gamma}\,d\tau = \int_S I \cdot E \cdot 2\pi\varrho\,da$$

or

$$R_2 i_2 = \int_S U\,\frac{I\,da}{i_2} \tag{19}$$

In the same manner we obtain from (5) and (6a):

$$L_2 i_2 = \int_S \varphi\,\frac{I\,da}{i_2} \tag{20}$$

$$M i_1 = \int_S \Phi\,\frac{I\,da}{i_2} \tag{21}$$

Now let us assume (see Fig. 3) that we divide the surface S into a large but finite number n of elements da, chosen so that the same current $I\,da$ passes through all of them. We then obtain n circular turns with radii ϱ between 0 and r, all carrying the same current $I\,da = i_2/n$.

We number the turns from 1 to n and denote by U_k, φ_k, and Φ_k the values of U, φ, and Φ, pertaining to turn No. k. Then

$$U_k = -\frac{d\varphi_k}{dt} - \frac{d\Phi_k}{dt} \tag{22}$$

and

$$\sum_1^n U_k = -\frac{d}{dt}\sum_1^n \varphi_k - \frac{d}{dt}\sum_1^n \Phi_k \tag{23}$$

Eqs. (19), (20), and (21) give us the approximate relations

$$R_2 i_2 = \frac{1}{n}\sum_1^n U_k;\quad L_2 i_2 = \frac{1}{n}\sum_1^n \varphi_k;\quad M i_1 = \frac{1}{n}\sum_1^n \Phi_k \tag{24 a, b, c}$$

Hence, from (23):

$$R_2 i_2 + L_2\,\frac{di_2}{dt} + M\,\frac{di_1}{dt} = 0 \tag{25 a}$$

i.e., the relation (15) already deduced.

Since the n turns all carry the same current i_2/n, they can be connected in series. We then obtain a coil with a certain resistance R_2^*, a certain inductance L_2^*, and a certain mutual inductance M^*. This coil will have a voltage between its ends of $\sum_1^n U_k$ and it will link the magnetic fluxes $\sum_1^n \varphi_k$ and $\sum_1^n \Phi_k$. Hence

$$R_2^* \frac{i_2}{n} = \sum_1^n U_k; \quad L_2^* \frac{i_2}{n} = \sum_1^n \varphi_k; \quad M^* i_1 = \sum_1^n \Phi_k \tag{26 a, b, c}$$

Comparing (24 a, b, c) and (26 a, b, c) we find

$$R_2^* = n^2 R_2; \quad L_2^* = n^2 L_2; \quad M^* = nM \tag{27 a, b, c}$$

From (23) follows

$$R_2^* \frac{i_2}{n} + L_2^* \frac{d}{dt} \frac{i_2}{n} + M^* \frac{di_1}{dt} = 0 \tag{25 b}$$

We can now summarize our investigation as follows:

Pursuing the eddy current paths, we divide the entire core into a great number of turns n, each turn carrying the same current i_2/n. We insulate the turns from each other by very thin insulating layers, cut them open, connect them in series, and short-circuit the coil thus obtained. This coil will have a resistance R_2^*, an inductance L_2^*, and a mutual inductance M^*. For the short-circuited coil the relation (25 b) is valid.

We could use this coil as our equivalent winding but for one serious disadvantage: in the limit $n \to \infty$ we would obtain a coil with infinitely large parameters, carrying an infinitely small current.

Therefore, it is more convenient to employ an entirely formal equivalent winding with the finite parameters

$$R_2 = \lim_{n \to \infty} \frac{R_2^*}{n^2}; \quad L_2 = \lim_{n \to \infty} \frac{L_2^*}{n^2}; \quad M = \lim_{n \to \infty} \frac{M^*}{n} \tag{28 a, b, c}$$

and carrying the finite current i_2. If we short-circuit this winding, eq. (25 a) = (15) holds. This equation differs from (25 b) only by the factor $1/n$.

The necessity of the condition that the geometric distribution of the eddy currents should be independent of time now becomes clear. This condition ensures that the n turns of our coil always carry equal currents, so that they can be connected in series, forming a fixed coil with *constant* parameters. Unfortunately this case is an exception; normally the eddy current pattern varies from one moment to the next, so that the configuration of the coil would have to change continuously. In other words, in most cases the equivalent winding cannot be used, because its parameters would vary with time. This is the reason why *the equivalent winding is of no value in a study of the relay breaking problem.*

On the other hand, the equivalent winding yields useful approximate results in such cases where the magnetic field produced by the eddy currents is small as compared with the main field produced by the current in a magnetizing wire winding. This latter field can then be considered as the only cause of the eddy currents, which can therefore be calculated from a homogeneously distributed magnetic flux varying with time. The parameters R_2, L_2, and M will then be constant in time and easy to compute.

If the eddy current magnetic field, although not small, has but little influence on the primary current and on the total magnetic flux, the same method can be reckoned upon to yield acceptable values of these two quantities. We shall make use of this when treating in the next chapter the relay closing problem. In this case we cannot expect very good results in the first moments where the eddy currents cause a quite large rate of increase of the primary current.

CHAPTER 3

Parameters of the eddy current equivalent winding of a relay on closing

Distribution along the core of the primary current magnetic flux

A constant primary current i_1 gives rise to a magnetic flux Φ_1 in core and yoke, the magnitude of which, on account of the magnetic leakage flux φ_λ, varies with the distance x from the heel piece end. As shown in Part 1, the function $\Phi_1 = \Phi_1(x)$ can with sufficient accuracy be approximated with a parabola (Part 1, eqs. (28) and (20 a)).

We introduce the quantities

$$a = \frac{\Sigma_0}{S_\lambda}; \quad b = \frac{\Sigma_1}{S_\lambda} \tag{29 a, b}$$

$$\eta = \sqrt{\frac{S}{S_\lambda}}; \quad \eta' = \sqrt{\frac{S'}{S_\lambda}}; \quad \eta'' = \sqrt{\frac{S''}{S_\lambda}} \tag{30 a, b, c}$$

$$g^2 = \eta^2 \frac{ab \dfrac{\sinh \eta}{\eta} + a \cosh \eta + b}{\eta^2 \dfrac{\sinh \eta}{\eta} + b(\cosh \eta - 1)} \tag{31 a}$$

Putting

$$\frac{x}{l} = u$$

we can write the results of Part 1:

$$\Phi_1(x) = \Phi_1(0)\,\lambda(u) \tag{32a}$$

with

$$\lambda(u) = 1 + au - \frac{1}{2}\,g^2u^2 \tag{32b}$$

and

$$\Phi_1(0) = \frac{N_1\,i_1/S_\lambda}{g^2 + \eta^2} \tag{32c}$$

Generally $\eta^2 \ll 1$. Then

$$g^2 \approx \frac{a + b + ab + \frac{1}{2}\,\eta^2 a\left(1 + \frac{1}{3}\,b\right)}{1 + \frac{1}{2}\,b + \frac{1}{6}\,\eta^2\left(1 + \frac{1}{4}\,b\right)} \tag{31b}$$

In our subsequent deductions the parabolic function $\lambda(u)$ gives rise to the following easily calculated average values:

$$m(u) = \int_0^u \lambda(u)\,du = u(1 + au/2 - g^2u^2/6) \tag{33a}$$

$$n(u) = 2\int_0^u m(u)\,du = u^2(1 + au/3 - g^2u^2/12) \tag{33b}$$

$$m_1 = m(1) \qquad = 1 + a/2 - g^2/6 \tag{34a}$$

$$n_1 = n(1) \qquad = 1 + a/3 - g^2/12 \tag{34b}$$

$$p_1 = 3\int_0^1 n(u)\,du \qquad = 1 + a/4 - g^2/20 \tag{34c}$$

$$q_1 = \int_0^1 \lambda(u)^2\,du \qquad = 1 + a - (g^2 - a^2)/3 - ag^2/4 + g^4/20 \tag{34d}$$

$$r_1 = \int_0^1 (1 + au)\,\lambda(u)\,du = 1 + a - (g^2 - 2a^2)/6 - ag^2/8 \tag{34e}$$

$$t_1 = 3\int_0^1 \lambda(u)\,n(u)\,du \qquad = 1 + a - (7g^2/4 - a^2)/5 - ag^2/8 + g^4/56 \tag{34f}$$

The eddy currents

In calculating the eddy currents, we proceed as stated in Chapter 2, *i.e.*, neglecting their own magnetic flux, we assume that they wholly result from the primary flux caused by the current i_1 through the magnetizing winding. This flux is uniformly distributed across core and yoke and varies parabolically along them; it changes with time in the same way as i_1.

Hence, it can be written

$$\Phi_1(x, t) = \lambda(u) f(t) \tag{35}$$

where the function $f(t)$ need not be specified.

1) The eddy currents in the core.

Applying Faraday's law to a circular path of radius ϱ, we obtain

$$2\pi\varrho \frac{I_2'(\varrho, x)}{\gamma'} = -\frac{d}{dt}\left\{\frac{\varrho^2}{r^2}\,\Phi_1(x, t)\right\} \tag{36a}$$

or, introducing $\Phi_1(x, t)$ from (35) and putting $-df(t)/dt = g(t)$:

$$I_2'(\varrho, x) = \frac{\gamma'}{2\pi r}\,\frac{\varrho}{r}\,\lambda(u)\,g(t) \tag{37a}$$

For the entire circulating current in the core we obtain

$$i_2' = \int\limits_{x=0}^{l}\int\limits_{\varrho=0}^{r} I_2'\,d\varrho\,dx = \frac{m_1 l}{4}\,\frac{\gamma'}{\pi}\,g(t) \tag{38a}$$

2) The eddy currents in the yoke.

Here we apply Faraday's law to a rectangular path of length a and width $2y$:

$$2a\,\frac{I_2''(y, x)}{\gamma''} = -\frac{d}{dt}\left\{\frac{2y}{c}\,\Phi_1(x, t)\right\} \tag{36b}$$

Thus,

$$I_2''(y, x) = \frac{\gamma''}{2a}\,\frac{y}{c/2}\,\lambda(u)\,g(t) \tag{37b}$$

and the entire circulating current

$$i_2'' = \int\limits_{x=0}^{l}\int\limits_{y=0}^{c/2} I_2''\,dy\,dx = \frac{m_1 l}{4}\,\frac{c/2}{a}\,\gamma'' g(t) \tag{38b}$$

3) The eddy current densities, expressed in the entire circulating current.

For the entire circulating current—by definition also the current in the equivalent winding—we obtain

$$i_2 = i_2' + i_2'' = \frac{m_1 l}{4}\left(\frac{\gamma'}{\pi} + \gamma''\,\frac{c/2}{a}\right) g(t) \tag{39}$$

We introduce the two quantities

$$\delta' = \frac{\gamma'/\pi}{\gamma'/\pi + \gamma'' c/2a}; \qquad \delta'' = \frac{\gamma'' c/2a}{\gamma'/\pi + \gamma'' c/2a} \tag{40 a, b}$$

δ' and δ'' show how the current i_2 is divided between core and yoke:

$$i_2' = \delta' i_2; \quad i_2'' = \delta'' i_2; \quad \delta' + \delta'' = 1 \tag{41 a, b, c}$$

Employing now (37 a, b), (39), and (40 a, b), we obtain the following final expressions for the eddy current densities:

$$I_2'(\varrho, x) = \frac{2\delta'}{m_1 l r} \frac{\varrho}{r} \lambda(u) \cdot i_2 \tag{42 a}$$

$$I_2''(y, x) = \frac{2\delta''}{m_1 l c/2} \frac{y}{c/2} \lambda(u) \cdot i_2 \tag{42 b}$$

The eddy current magnetic field

The calculation of L_2, M, and M_{21} requires that we know the magnetic field of the eddy currents. We need (see *Fig. 4*) the following two quantities:

$\varphi'(\varrho, x) = \int_0^\varrho \mu' H'(\varrho, x)\, 2\pi\varrho\, d\varrho =$ the eddy current magnetic flux in the core, linked with a circular eddy current path with the radius ϱ, situated at the distance x from the heel piece end.

$\varphi''(y, x) = \int_0^y \mu'' H''(y, x)\, 2a\, dy =$ the eddy current magnetic flux in the yoke, linked with a rectangular eddy current path with the length a and the width $2y$, situated at the distance x from the heel piece end.

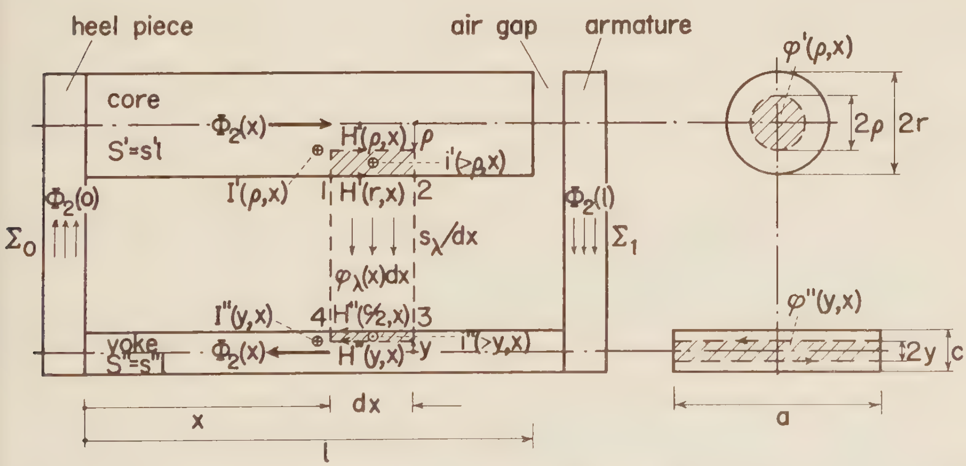

Fig. 4. To the calculation of the eddy current magnetic fluxes.

We also introduce

$\Phi_2(x)$ = the total eddy current flux in core or yoke at the distance x from the heel piece end.

Evidently

$$\Phi_2(x) = \varphi'(r, x) = \varphi''(c/2, x) \tag{43}$$

In Part 1, Chapter 2, and again in Part 2, Chapter 3, are given the relations which govern the magnetic fluxes Φ and φ_λ produced by a constant current i_1 through a magnetizing wire winding covering the whole length of the core. These relations consist of two differential equations (Part 2, eqs. (62) and (63)) and two end conditions (Part 2, eqs. (64) and (65)).

In our present case, where the fluxes are produced by the eddy currents, the first differential equation, expressing the flux continuity, can be taken over unchanged:

$$\varphi_\lambda = -\frac{d\Phi_2}{dx} \tag{44}$$

The same holds good for the end conditions:

$$\Sigma_0 \Phi_2(0) + s_\lambda \varphi_\lambda(0) = 0; \quad \Sigma_1 \Phi_2(l) - s_\lambda \varphi_\lambda(l) = 0 \tag{45 a, b}$$

In Part 2 we used the symbols Φ and φ_λ. In (44) and (45 a, b) we have replaced Φ by Φ_2 but for simplicity we have retained φ_λ although in the present case this flux also derives from the eddy currents.

There remains the second differential equation,

$$s\Phi + s_\lambda \frac{d\varphi_\lambda}{dx} = \frac{N_1 i_1}{l} \tag{46}$$

obtained as an application of Ampère's law. This equation implies a uniformly distributed flux Φ, caused by a current i_1 circulating on the core surface. In our present case this condition is not fulfilled, whence the equation cannot be taken over unchanged.

In fact, the flux Φ_2 is caused by eddy currents circulating *within* core and yoke, and the flux distribution across a section of core or yoke is strongly non-uniform. We shall show that (46) must be replaced by (51).

We apply Ampère's law to the rectangular path 1234 in Fig. 4, situated in the interspace between core and yoke. No current being linked with the path, we obtain

$$s_\lambda \varphi_\lambda(x+dx) - s_\lambda \varphi_\lambda(x) + \{H'(r, x) + H''(c/2, x)\}\, dx = 0$$

or

$$H'(r, x) + H''(c/2, x) = -s_\lambda \frac{d\varphi_\lambda}{dx} \tag{47}$$

where $H'(r, x)$ and $H''(c/2, x)$ are the magnetic intensities at the surfaces of core and yoke.

We next apply Ampère's law to the two rectangular paths enclosing the hatched areas within the core and the yoke, and obtain

$$H'(\varrho, x) - H'(r, x) = i'(>\varrho, x) \tag{48 a}$$

$$H''(y, x) - H''(c/2, x) = i''(>y, x) \tag{48 b}$$

where $i'(>\varrho, x)$ = the total circulating current per unit core length outside a cylinder with radius ϱ.

$i''(>y, x)$ = the total circulating current per unit yoke length outside a rectangular strip with the transverse dimensions a and $2\,y$.

Using (42 a, b), we immediately calculate

$$i'(>\varrho, x) = \int_{\varrho}^{r} I_2'(\varrho, x)\, d\varrho = \frac{\delta'}{m_1 l}\left\{1 - \left(\frac{\varrho}{r}\right)^2\right\} \lambda(u) \cdot i_2 \tag{49 a}$$

$$i''(>y, x) = \int_{y}^{c/2} I_2''(y, x)\, dy = \frac{\delta''}{m_1 l}\left\{1 - \left(\frac{y}{c/2}\right)^2\right\} \lambda(u) \cdot i_2 \tag{49 b}$$

We now turn to the two expressions for Φ_2 given in (43):

$$\Phi_2(x) = \int_0^r \mu' H'(\varrho, x)\, 2\pi\varrho\, d\varrho = \int_0^{c/2} \mu'' H''(y, x)\, 2a\, dy$$

Introducing here $H'(\varrho, x)$ and $H''(y, x)$ from (48 a, b), and making use of (49 a, b), we find

$$H'(r, x) = s'\Phi_2(x) - \frac{\delta'}{2m_1 l}\, \lambda(u) \cdot i_2 \tag{50 a}$$

$$H''(c/2, x) = s''\Phi_2(x) - \frac{2\delta''}{3m_1 l}\, \lambda(u) \cdot i_2 \tag{50 b}$$

Introducing these expressions in (47) and observing the relations $s' + s'' = s$, $\delta' + \delta'' = 1$, we obtain the following equation, replacing (46):

$$s\Phi_2 + s_\lambda \frac{d\varphi_\lambda}{dx} = \frac{1 + \delta''/3}{2m_1 l}\, \lambda(u) \cdot i_2 \tag{51}$$

Our next task is to find $\Phi_2 - \Phi_2(x)$ from (44), (45 a, b), and (51). Comparing (51) and (46), we see that for the purpose of computing Φ_2 we can still assume a current on the core surface. This current, however, is not independent of x as in Parts 1 and 2 but varies parabolically along the core.

In principle, an exact integration of the system (44, 51) offers no difficulties. Still, we prefer an approximate method, which, while being sufficiently accurate, involves much simpler calculations.

The term $s\Phi_2$ in (51) represents the magnetic voltage drop per unit length of core + yoke, a quantity which is generally quite small. We can therefore, without substantial loss of accuracy, replace it by an approximate expression, which is known to us. With $s\Phi_2$ known, an integration immediately gives us φ_λ from (51), and by a subsequent integration we compute Φ_2 (a better value than the one used in $s\Phi_2$!) from (44). Finally, the end conditions (45 a, b) give us the two integration constants.

As our approximate value of $s\Phi_2$ we choose the value, obtained by replacing the parabolically distributed current in (51) with a constant current, giving the same total mmf. We are thus brought back to the relation (46) so that Φ_2 follows at once from the results of Parts 1 and 2.

We denote the constant current per unit core length by i_2^*. Hence, equalizing the mmf's:

$$i_2^* l = \int_0^l \frac{1+\delta''/3}{2m_1 l} \lambda(u) i_2 \, dx = \frac{1+\delta''/3}{2} i_2$$

i.e.

$$i_2^* = \frac{1+\delta''/3}{2l} i_2 \tag{52}$$

Substituting now in (32 c) $\Phi_2(0)$ for $\Phi_1(0)$ and $i_2^* l$ for $N_1 i_1$, we find

$$S\Phi_2(0) = i_2^* l \frac{\eta^2}{g^2+\eta^2}$$

and hence, observing that $S = ls$:

$$s\Phi_2(x) = s\Phi_2(0)\,\lambda(u) = \frac{1+\delta''/3}{2l} \frac{\eta^2}{g^2+\eta^2} \lambda(u) \cdot i_2 \tag{53}$$

Using this expression for $s\Phi_2$, we obtain from (51):

$$s_\lambda \frac{d\varphi_\lambda}{dx} = \lambda(u) \cdot i_2^{**} \tag{54}$$

with

$$i_2^{**} = \frac{1+\delta''/3}{2m_1 l} \left(1 - \frac{m_1 \eta^2}{g^2+\eta^2}\right) i_2 \tag{55}$$

Integrating (54), while observing (33 a), we find

$$\varphi_\lambda(x) = \varphi_\lambda(0) + \frac{i_2^{**} l}{s_\lambda} m(u) \tag{56}$$

We now integrate (44), making use of (56), (33 b), and the relation $s_\lambda = lS_\lambda$. It follows that

$$\Phi_2(x) = \Phi_2(0) - x\varphi_\lambda(0) - \frac{i_2^{**} l}{2S_\lambda} n(u) \tag{57}$$

The unknown constants $\Phi_2(0)$ and $\varphi_\lambda(0)$ are determined by (45 a, b). One finds that (57) can be written

$$\Phi_2(x) = \Phi_2(0)\,(1 + au) - \frac{i_2^{**} l}{2S_\lambda}\, n(u) \tag{58}$$

with

$$\Phi_2(0) = \beta\, \frac{i_2^{**} l}{S_\lambda}; \quad \beta = \frac{m_1 + \frac{1}{2}\, bn_1}{a + b + ab} \tag{59 a, b}$$

Knowing Φ_2, we can now determine the magnetic field intensities $H'(\varrho, x)$ and $H''(y, x)$. The eqs. (48 a, b) together with (49 a, b) and (50 a, b) give us

$$H'(\varrho, x) = s'\,\Phi_2(x) + \frac{\delta'}{2m_1 l}\left\{1 - 2\left(\frac{\varrho}{r}\right)^2\right\}\lambda(u)\cdot i_2 \tag{60 a}$$

$$H''(y, x) = s''\,\Phi_2(x) + \frac{\delta''}{3m_1 l}\left\{1 - 3\left(\frac{y}{c/2}\right)^2\right\}\lambda(u)\cdot i_2 \tag{60 b}$$

Introducing $\Phi_2(x)$ from (58), and paying attention to (55) and (59 a, b), we finally obtain

$$H'(\varrho, x) = \frac{1}{2m_1 l}\left\{\eta'^2\left(1 + \frac{\delta''}{3}\right)\left(1 - \frac{m_1\eta^2}{g^2 + \eta^2}\right)\left[\beta\,(1 + au) - \frac{1}{2}\, n(u)\right] + \right.$$
$$\left. + \delta'\left[1 - 2\left(\frac{\varrho}{r}\right)^2\right]\right\}\lambda(u)\cdot i_2 \tag{61 a}$$

$$H''(y, x) = \frac{1}{2m_1 l}\left\{\eta''^2\left(1 + \frac{\delta''}{3}\right)\left(1 - \frac{m_1\eta^2}{g^2 + \eta^2}\right)\left[\beta\,(1 + au) - \frac{1}{2}\, n(u)\right] + \right.$$
$$\left. + \frac{2\delta''}{3}\left[1 - 3\left(\frac{y}{c/2}\right)^2\right]\right\}\lambda(u)\cdot i_2 \tag{61 b}$$

From the expressions (61 a, b) we compute without difficulty the fluxes $\varphi'(\varrho, x)$ and $\varphi''(y, x)$. The result is

$$\varphi'(\varrho, x) = \left\{\frac{1 + \delta''/3}{2m_1 S_\lambda}\left(1 - \frac{m_1\eta^2}{g^2 + \eta^2}\right)\left(\frac{\varrho}{r}\right)^2\left[\beta\,(1 + au) - \frac{1}{2}\, n(u)\right] + \right.$$
$$\left. + \frac{\delta'}{2m_1 S'}\left(\frac{\varrho}{r}\right)^2\left[1 - \left(\frac{\varrho}{r}\right)^2\right]\lambda(u)\right\}\cdot i_2 \tag{62 a}$$

$$\varphi''(y, x) = \left\{\frac{1 + \delta''/3}{2m_1 S_\lambda}\left(1 - \frac{m_1\eta^2}{g^2 + \eta^2}\right)\frac{y}{c/2}\left[\beta\,(1 + au) - \frac{1}{2}\, n(u)\right] + \right.$$
$$\left. + \frac{\delta''}{3m_1 S''}\,\frac{y}{c/2}\left[1 - \left(\frac{y}{c/2}\right)^2\right]\lambda(u)\right\}\cdot i_2 \tag{62 b}$$

As a check we compute $\Phi_2(x)$ from both expressions:

$$\Phi_2(x) = \varphi'(r, x) = \varphi''(c/2, x) =$$
$$= \frac{1 + \delta''/3}{2m_1 S_\lambda}\left(1 - \frac{m_1 \eta^2}{g^2 + \eta^2}\right)\left[\beta\,(1 + au) - \frac{1}{2}\,n(u)\right]\cdot i_2 \tag{63}$$

and observe that the same value of $\Phi_2(x)$ follows from (58), combined with (55) and (59 a, b).

The resistance of the equivalent winding

From eq. (4) follows

$$R_2 = \int\limits_{x=0}^{l}\int\limits_{\varrho=0}^{r}\frac{(I_2'/i_2)^2}{\gamma'}\,2\pi\varrho\,d\varrho\,dx + \int\limits_{x=0}^{l}\int\limits_{y=-c/2}^{+c/2}\frac{(I_2''/i_2)^2}{\gamma''}\,a\,dy\,dx \tag{64}$$

Taking I_2', I_2'' from (42 a, b) and performing the integrations, we easily find

$$R_2 = \frac{q_1}{m_1^2 l}\left(\frac{2\pi}{\gamma'}\,\delta'^2 + \frac{16a/3c}{\gamma''}\,\delta''^2\right) \tag{65}$$

The inductance of the equivalent winding

According to eq. (5) we have

$$L_2 = L_2' + L_2'' \tag{66}$$

with

$$L_2' = \frac{1}{i_2^2}\int\limits_{\text{core}}\mathbf{I}_2'\cdot\mathbf{A}_2'\,d\tau'; \qquad L_2'' = \frac{1}{i_2^2}\int\limits_{\text{yoke}}\mathbf{I}_2''\cdot\mathbf{A}_2''\,d\tau'' \tag{67 a, b}$$

Reasoning as when deducing (20), we obtain

$$L_2' = \int\limits_{x=0}^{l}\int\limits_{\varrho=0}^{r}\frac{I_2'}{i_2}\,\frac{\varphi'}{i_2}\,d\varrho\,dx; \quad L_2'' = \int\limits_{x=0}^{l}\int\limits_{y=0}^{c/2}\frac{I_2''}{i_2}\,\frac{\varphi''}{i_2}\,dy\,dx \tag{68 a, b}$$

With the aid of (42 a, b) and (62 a, b) this gives us

$$L_2' = \frac{\delta'}{12 m_1^2 S_\lambda}\left\{\left(1 + \frac{\delta''}{3}\right)\left(1 - \frac{m_1\eta^2}{g^2 + \eta^2}\right)\left(3\beta r_1 - \frac{1}{2}t_1\right) + q_1\,\frac{\delta'}{\eta'^2}\right\} \tag{69 a}$$

$$L_2'' = \frac{\delta''}{12 m_1^2 S_\lambda}\left\{\frac{4}{3}\left(1 + \frac{\delta''}{3}\right)\left(1 - \frac{m_1\eta^2}{g^2 + \eta^2}\right)\left(3\beta r_1 - \frac{1}{2}t_1\right) + \frac{16 q_1 \delta''}{15\,\eta''^2}\right\} \tag{69 b}$$

and, if we make use of $\delta' + \delta'' = 1$:

$$L_2 = \frac{1}{12 m_1^2 S_\lambda}\left\{\left(1 + \frac{\delta''}{3}\right)^2\left(1 - \frac{m_1\eta^2}{g^2 + \eta^2}\right)\left(3\beta r_1 - \frac{1}{2}t_1\right) + q_1\left(\frac{\delta'^2}{\eta'^2} + \frac{16\delta''^2}{15\,\eta''^2}\right)\right\} \tag{70}$$

The mutual inductance between the equivalent winding and the primary winding

For the mutual inductance M we derive two different expressions, one from (6 a) and the other from (6 b).

Taking first (6 a), we proceed as when calculating L_2 and find

$$M = M' + M'' \tag{71}$$

with

$$M' = \int_{x=0}^{l} \int_{\varrho=0}^{r} \frac{I_2'}{i_2} \frac{\Phi_1'}{i_1} \, d\varrho \, dx; \quad M'' = \int_{x=0}^{l} \int_{y=0}^{c/2} \frac{I_2''}{i_2} \frac{\Phi_1''}{i_1} \, dy \, dx \tag{72 a, b}$$

Here

$\Phi_1' = \Phi_1'(\varrho, x) =$ the primary current magnetic flux in the core, linked with a circular eddy current path of radius ϱ situated at the distance x from the heel piece end.

$\Phi_1'' = \Phi_1''(y, x) =$ the primary current magnetic flux in the yoke, linked with a rectangular eddy current path of length a and width $2y$ situated at the distance x from the heel piece end.

Introducing the total primary current flux

$$\Phi_1(x) = \Phi_1'(r, x) = \Phi_1''(c/2, x)$$

we have

$$\Phi_1'(\varrho, x) = \left(\frac{\varrho}{r}\right)^2 \Phi_1(x); \quad \Phi_1''(y, x) = \frac{y}{c/2} \Phi_1(x) \tag{73 a, b}$$

with $\Phi_1(x)$ according to (32 a, b, c).

We now integrate (72 a, b). The result is

$$M' = \frac{N_1}{2m_1 S_\lambda} \frac{\delta'}{g^2 + \eta^2} q_1; \quad M'' = \frac{N_1}{2m_1 S_\lambda} \frac{4\delta''/3}{g^2 + \eta^2} q_1 \tag{74 a, b}$$

and

$$M = \frac{N_1}{2m_1 S_\lambda} \left(1 + \frac{\delta''}{3}\right) \frac{q_1}{g^2 + \eta^2} \tag{75}$$

We pass on to (6 b). This relation has already been transformed to (7 b), which can be written

$$M = \frac{N_1}{l} \int_{x=0}^{l} \frac{\varphi'(r, x)}{i_2} \, dx \tag{76}$$

Introducing $\varphi'(r, x)$ from (63), we find

$$M = \frac{N_1}{2m_1 S_\lambda} \left(1 + \frac{\delta''}{3}\right) \left(1 - \frac{m_1 \eta^2}{g^2 + \eta^2}\right) \left[\beta\left(1 + \frac{a}{2}\right) - \frac{1}{6} p_1\right] \tag{77}$$

Generally, (75) and (77) lead to somewhat different values of M. This is due to the fact that the expression (63) for the eddy current flux $\varphi'(r, x)$ depends on the approximate value of $s\Phi_2$ which we used in (51). For normal values of $S = sl$, however, (75) and (77) yield very nearly the same numerical result. This confirms that our approximation of $s\Phi_2$ was permissible.

It is not difficult to see from (51) that in two cases, for $S = 0$ and for $S_\lambda = \infty$, our method of solution is exact. By comparing (75) and (77) one also finds that in both of these cases the two expressions for M coincide.

The mutual inductance between the equivalent winding and a fictitious winding of one turn at the armature end

Eq. (16) immediately gives us

$$M_{21} = \frac{\varphi'(r, l)}{i_2} \tag{78}$$

where $\varphi'(r, l)$ follows from (63). It ensues

$$M_{21} = \frac{1}{2m_1 S_\lambda}\left(1 + \frac{\delta''}{3}\right)\left(1 - \frac{m_1\eta^2}{g^2+\eta^2}\right)\left[\beta\,(1+a) - \frac{1}{2}\,n_1\right] \tag{79}$$

CHAPTER 4

The closing of relays without magnetic leakage and with eddy currents in the core only

In this chapter we study a simplified case, accessible to an exact treatment. It allows us to judge the usefulness of the equivalent winding solution by comparing it with the exact solution.

The exact solution

a) *Expansion theorem solution for large t.*

As mentioned in Part 3, a paper by I. Herlitz[5], published in 1920, implicitly contains the solution of the eddy current problem for a relay with eddy currents in the core only, and without magnetic leakage. Making use of Heaviside's expansion theorem, Herlitz could express his quantities by means of exponential function series valid for not too small values of

time t. His expression for the magnetic flux, written with the symbols of our telephone relay problem, is presented in Part 3, eq. (29).

We give below the expansion theorem solutions for the following four quantities:

$i_1 = i_1(t) =$ the primary current.

$\Phi = \Phi(t) =$ the magnetic flux.

$H = H(t, \varrho) =$ the magnetic intensity in the core.

$I = I(t, \varrho) =$ the eddy current density in the core.

($\varrho =$ distance from the core axis.)

The subsequent notations are used:

$$\tau = r^2 \gamma' \mu'; \quad K = \frac{S'' + \Sigma_0 + \Sigma_1}{S'}$$

and

$$\Delta_i = \Delta(x_i) = \frac{x_i}{2} \frac{J_0(x_i)}{J_1(x_i)}$$

where J_0 and J_1 are Bessel functions and x_i ($i = 1, 2, 3, \ldots$) are the (real and positive) roots of

$$\Delta(x) - \frac{N_1^2}{R_1 S' \tau} x^2 + K = 0$$

The solutions are:

$$\frac{i_1(t)}{i_{1\infty}} = 1 - \sum_{i=1}^{\infty} \frac{\Delta_i + K}{\Delta_i^2 + x_i^2/4 + K} e^{-\frac{t}{\tau/x_i^2}}; \quad i_{1\infty} = i_1(\infty) = \frac{E_1}{R_1} \tag{80a}$$

$$\frac{\Phi(t)}{\Phi_\infty} = 1 - \sum_{i=1}^{\infty} \frac{1 + K}{\Delta_i^2 + x_i^2/4 + K} e^{-\frac{t}{\tau/x_i^2}}; \quad \Phi_\infty = \Phi(\infty) = \frac{N_1 i_{1\infty}}{S'(1 + K)} \tag{80b}$$

$$\frac{H(t, \varrho)}{H_\infty} = 1 - \sum_{i=1}^{\infty} \frac{(1 + K)\Delta_i}{\Delta_i^2 + x_i^2/4 + K} \frac{J_0\left(\frac{\varrho}{r} x_i\right)}{J_0(x_i)} e^{-\frac{t}{\tau/x^2}}; \quad H_\infty = H(\infty, \varrho) = \frac{N_1 i_{1\infty}}{l(1 + K)} \tag{80c}$$

$$\frac{I(t, \varrho)}{I_0} = \sum_{i=1}^{\infty} \frac{\Delta_i + K}{\Delta_i^2 + x_i^2/4 + K} \frac{J_1\left(\frac{\varrho}{r} x_i\right)}{J_1(x_i)} e^{-\frac{t}{\tau/x_i^2}}; \quad I_0 = I(0, r) = \frac{\gamma' E_1}{2\pi r N_1} \tag{80d}$$

b) *Series solutions for small t.*

For small values of t we employ series solutions in powers of $t^{1/2}$, obtained by developing the operators in powers of $p^{-1/2}$. The calculations are similar to those given in Part 3 for a relay *with* magnetic leakage (*cf.* Part 3, eq. (124), for the armature end flux; with leakage the series is in powers of $t^{1/4}$).

The following expressions for the primary current i_1 and the magnetic flux Φ contain terms up to and including t^3 in i_1 and Φ/t:

$$\frac{i_1(t)}{i_{1\infty}} = \frac{2k}{\sqrt{\pi}}\left(\frac{t}{\tau}\right)^{1/2}\left[1+\frac{\sqrt{\pi}A_1}{2}\left(\frac{t}{\tau}\right)^{1/2}+\frac{2A_2}{3}\frac{t}{\tau}+\frac{\sqrt{\pi}A_3}{4}\left(\frac{t}{\tau}\right)^{3/2}+\right.$$
$$\left.+\frac{4A_4}{15}\left(\frac{t}{\tau}\right)^{2}+\frac{\sqrt{\pi}A_5}{12}\left(\frac{t}{\tau}\right)^{5/2}+\ldots\right];\quad i_{1\infty}=i_1(\infty)=\frac{E_1}{R_1} \tag{81 a}$$

$$\frac{\Phi(t)}{\Phi_0'} = t\left\{1-\frac{4k}{3\sqrt{\pi}}\left(\frac{t}{\tau}\right)^{1/2}\left[1+\frac{3\sqrt{\pi}A_1}{8}\left(\frac{t}{\tau}\right)^{1/2}+\frac{2A_2}{5}\frac{t}{\tau}+\frac{\sqrt{\pi}A_3}{8}\left(\frac{t}{\tau}\right)^{3/2}+\right.\right.$$
$$\left.\left.+\frac{4A_4}{35}\left(\frac{t}{\tau}\right)^{2}+\frac{\sqrt{\pi}A_5}{32}\left(\frac{t}{\tau}\right)^{5/2}+\ldots\right]\right\};\quad \Phi_0'=\left[\frac{d\Phi}{dt}\right]_{t=0}=\frac{E_1}{N_1} \tag{81 b}$$

In these formulæ

$$k=\frac{1}{2}\,\frac{R_1 S'\tau}{N_1^2}$$

and

$$\begin{aligned}
A_1 &= -\,(k-q)\\
A_2 &= \quad k^2-2qk+3/8\\
A_3 &= -\,(k^3-3qk^2+(q^2+3/4)k-3/8)\\
A_4 &= \quad (k^4-4qk^3+q^2+3/8)k^2-(q+1)3k/4+63/128\\
A_5 &= -\,(k^5-5qk^4+(q^2+1/4)6k^3-(q^3+9q/4+9/8)k^2+(q+3/2)3k/4-27/32
\end{aligned}$$

with

$$q=\frac{1}{2}+2K=\frac{1}{2}+2\,\frac{S''+\Sigma_0+\Sigma_1}{S'}$$

The equivalent winding parameters

Our present, simplified problem is characterized by the conditions

$$S_\lambda=\infty;\ \gamma''=0$$

They give rise to the following numerical values:

$$a = b = \eta^2 = \eta'^2 = \eta''^2 = g = 0$$

$$m_1 = n_1 = p_1 = q_1 = r_1 = t_1 = 1$$

$$\delta' = 1; \quad \delta'' = 0$$

Introducing the total magnetic reluctance

$$\Sigma = \Sigma_0 + \Sigma_1 + S' + S'' = \Sigma_0 + \Sigma_1 + S$$

and observing the relations

$$\lim_{S_\lambda \to \infty} \frac{\eta^2}{g^2 + \eta^2} = \frac{S}{\Sigma}; \qquad \lim_{S_\lambda \to \infty} \frac{\beta}{S_\lambda} = \frac{1}{\Sigma_0 + \Sigma_1}$$

we find from (65), (70), (75) or (77), and (79) the following values of the equivalent winding parameters:

$$R_2 = \frac{2\pi}{\gamma' l}; \qquad L_2 = \frac{1}{12 S'}\left(1 + \frac{3S'}{\Sigma}\right) \tag{82 a, b}$$

$$M = \frac{N_1}{2\Sigma}; \quad M_{21} = \frac{1}{2\Sigma} \tag{82 c, d}$$

We add the parameters of the primary winding:

$$L_1 = \frac{N_1^2}{\Sigma}; \quad M_{11} = \frac{N_1}{\Sigma} \tag{82 e, f}$$

The formulæ for primary current and armature end flux (Part 2, eqs. (44) and (46); see also the list of formulæ at the end of this paper) make use of the following quantities:

primary time constant

$$T_1 = \frac{L_1}{R_1} = \frac{N_1^2/R_1}{\Sigma} \tag{83 a}$$

secondary time constant

$$T_2 = \frac{L_2}{R_2} = \frac{\gamma' l}{24\pi S'}\left(1 + \frac{3S'}{\Sigma}\right) \tag{83 b}$$

time constant in the formula for the armature end flux

$$T_{21} = T_2 - \frac{M_{21}}{M_{11}}\frac{M}{R_2} = \frac{\gamma' l}{24\pi S'} \tag{83 c}$$

leakage factor

$$\sigma = 1 - \frac{M^2}{L_1 L_2} = \frac{1}{1 + 3S'/\Sigma} \tag{83 d}$$

Basic data of the quantitative study

The quantitative study of this and the next chapter refers to the relay already investigated in the preceding Parts 1, 2, and 3. The data of this relay were given in Part 3, Chapter 4 (for the geometric data see the second paragraph following the chapter heading; for the electric and magnetic data see Tables 2 and 6).

In short, the data refer to an Ericsson RAB relay with a Lancashire iron yoke and a Lancashire or silicon iron core, an air gap of 0.10 or 0.87 mm, and a winding of 10 000 turns with the resistance 560 Ω. In the present chapter we put the leakage reluctances $S_\lambda = \infty$.

Two series of permeability values are employed, *viz.*, the series of high values, corresponding to the mmf 300 At, already used in Part 3, and a series of low values, equal to 1/10 of the values of the first series.

In *Table 1* the different cases treated are specified and suitably designated.

Table 1. The numerical cases treated in Chapters 4 and 5.

Case	Core	Air gap mm	Relative permeability
L87(μ)	Lancashire	0.87	5 400
L10(μ)		0.10	3 400
S87(μ)	Silicon	0.87	4 500
S10(μ)		0.10	3 400
L87(μ/10)	Lancashire	0.87	540
L10(μ/10)		0.10	340
S87(μ/10)	Silicon	0.87	450
S10(μ/10)		0.10	340

Results

Fig. 5 gives primary current and magnetic flux *vs.* time in a case where the eddy current influence stands out clearly. The curves for actual relays with magnetic leakage have the same general appearance.

Without eddy currents both curves are exponential with the time constant 13.7 ms. With eddy currents the flux curve starts with an unchanged rate of increase. As compared to the exponential curve, it then slows down, remaining all the time below this curve. In other words: the eddy currents always cause a decrease of the flux value.

The current curve starts with an infinite rate of increase; we find from the power series development that $di_1/dt \approx$ const. $\times\ t^{-1/2}$ for very small values of t. The curve at first rises well above the exponential curve but after a while its rate of increase becomes so small that it cuts across the exponential curve, remaining subsequently below it.

Thus, the primary current curve displays a pronounced knee. In Part 2 we found the same phenomenon in relays with delaying slugs and, in its extreme, in the sleeved relays, where the current rises to its final value almost at once.

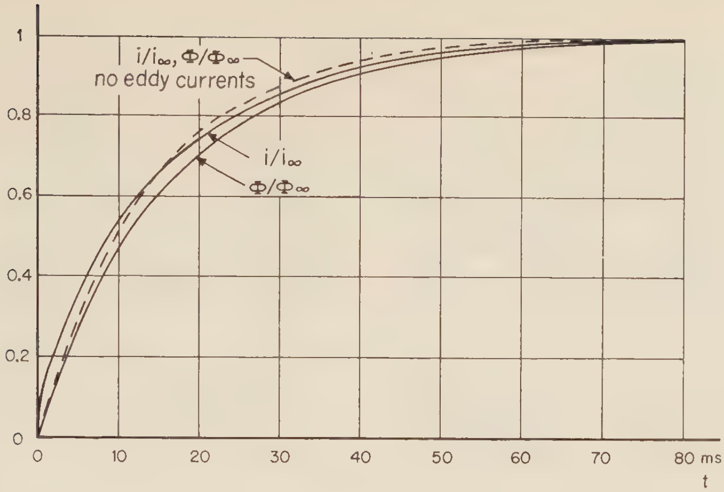

Fig. 5. Primary current and magnetic flux on closing a relay without magnetic leakage and with eddy currents in the core only. Solid curves: case L87(μ/10), Lancashire core, relative permeability of core material = 540, air gap 0.87 mm. Dotted curve: exponential curve, time constant 13.7 ms, valid for relay without eddy currents.

The effects of a change in the permeability value are of interest. An increased permeability of course leads to a larger time constant and hence to an exponential curve with a slower increase. Closer study (*cf.* Figs. 10 a to 13 a) shows that an increase of the permeability entails a decrease of the eddy current influence and a shift toward higher time values of the intersection between the current curve and the exponential curve.

The four pairs of figures, *Figs. 6 a, b to 9 a, b*, give an idea of the accuracy of the equivalent winding solutions. They all refer to the large air gap (0.87 mm), the one of primary interest at closing. The results for the small air gap (0.10 mm) are similar but somewhat more favorable as to the applicability of the equivalent winding.

Figs. 6 a, b refer to case L87(μ), the case with the most marked magnetic skin effect and the largest error in the equivalent winding solution. For four time values, 2.5, 10, 40, and 160 ms, Fig. 6 a gives the distribution across the core section of the magnetic intensity H (to the left), and the eddy current density I (to the right).

One sees that still at 160 ms the distribution of the magnetic flux across the core is extremely inhomogeneous and the eddy current density quite far from the linear increase from core axis to core surface required by our theory (*cf.* eq. (42 a)). Yet, as shown by Fig. 6 b, the equivalent winding solution gives reasonably accurate results except for the primary current at small times.

At the other end we have case S87(μ/10), Figs. 9 a, b, with a fairly small magnetic skin effect. Here the eddy current density follows the linear law much better, and the equivalent winding gives acceptable results down to quite small time values.

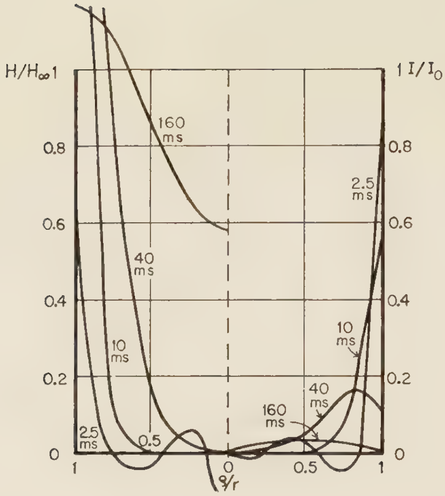

Fig. 6 a. Distribution across the core section of magnetic intensity and eddy current density on closing. No magnetic leakage. Eddy currents in core only. Case L87(μ).

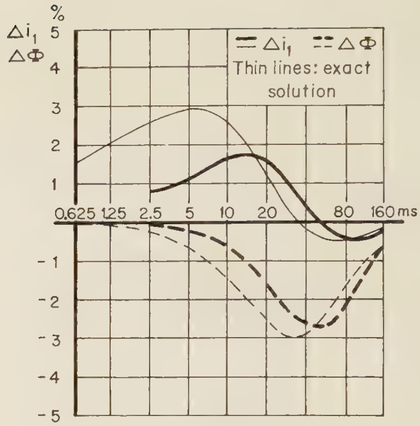

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Fig. 6 b. Comparison between the exact and the equivalent winding solutions for primary current and magnetic flux on closing. Deviations from values of relay without eddy currents, in percent of final steady state value. No magnetic leakage. Eddy currents in core only. Case L87(μ).

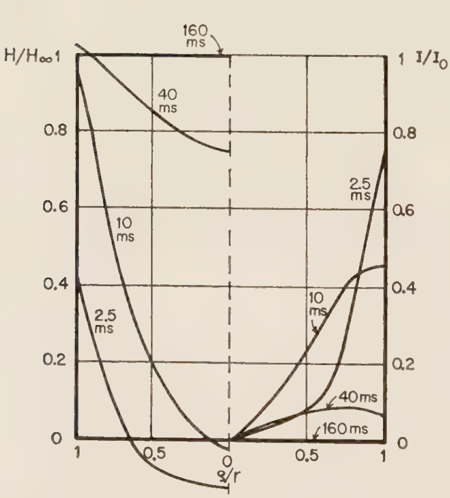

Fig. 7 a. Distribution across the core section of magnetic intensity and eddy current density on closing. No magnetic leakage. Eddy currents in core only. Case L87(μ/10).

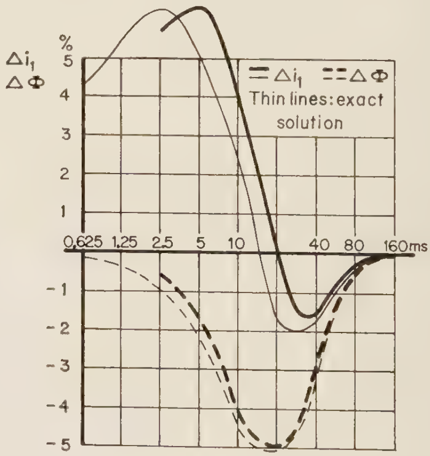

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Fig. 7 b. Comparison between the exact and the equivalent winding solutions for primary current and magnetic flux on closing. Deviations from values of relay without eddy currents, in percent of final steady state value. No magnetic leakage. Eddy currents in core only. Case L87(μ/10).

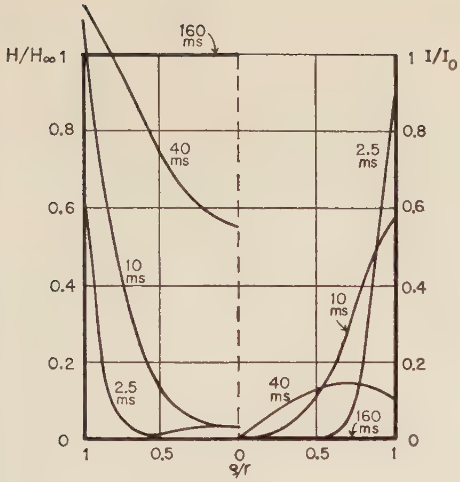

Fig. 8 a. Distribution across the core section of magnetic intensity and eddy current density on closing. No magnetic leakage. Eddy currents in core only. Case S87(μ).

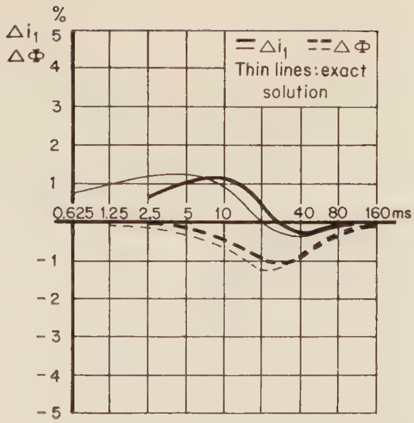

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Fig. 8 b. Comparison between the exact and the equivalent winding solutions for primary current and magnetic flux on closing. Deviations from values of relay without eddy currents, in percent of final steady state value. No magnetic leakage. Eddy currents in core only. Case S87(μ).

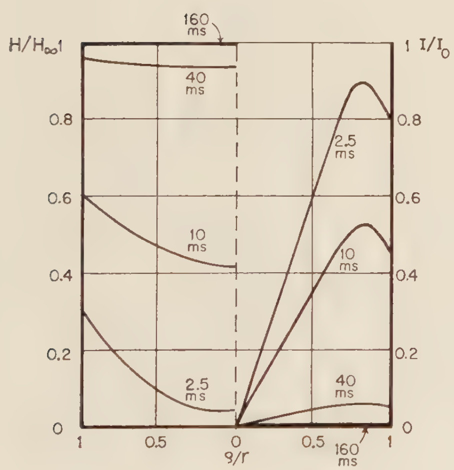

Fig. 9 a. Distribution across the core section of magnetic intensity and eddy current density on closing. No magnetic leakage. Eddy currents in core only. Case S87 $(\mu/10)$.

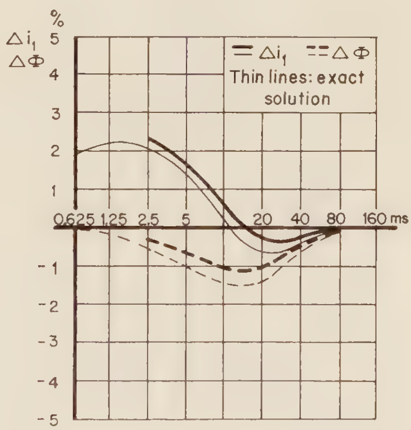

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Fig. 9 b. Comparison between the exact and the equivalent winding solutions for primary current and magnetic flux on closing. Deviations from values of relay without eddy currents, in percent of final steady state value. No magnetic leakage. Eddy currents in core only Case S87$(\mu/10)$.

The closing of relays with magnetic leakage and with eddy currents in core and yoke

Numerical results

We now turn to the study of actual relays with magnetic leakage and with eddy currents in both core and yoke. We employ the values of the leakage reluctance S_λ, given i Part 3, Chapter 4, Table 2. As in Part 3 (see Part 3, Chapter 4, Table 6 and the two paragraphs above the table), we take account of the eddy currents in the armature and the heel piece, as well as in those parts of the core which are not covered by the winding, by properly increasing the values of the core and yoke conductivities.

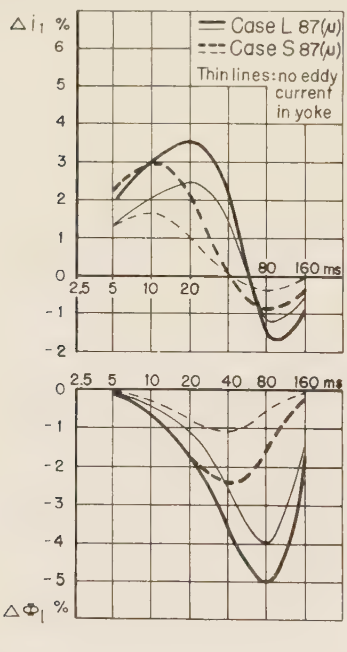

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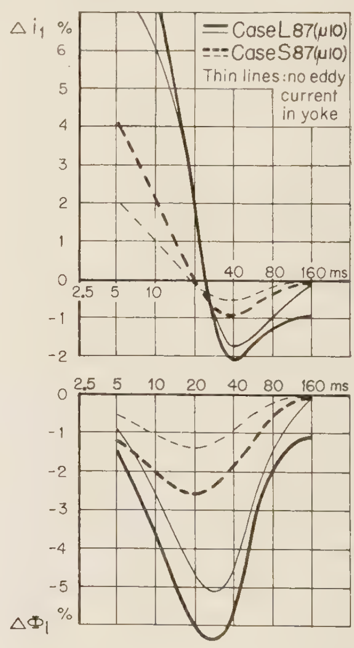
Fig. 10. Influence of eddy currents on primary current and armature end magnetic flux on closing. Deviations from values in relay without eddy currents, in percent of final steady state value. Relay with magnetic leakage. Cases L87(μ) and S87(μ).



X 11205

Fig. 11. Influence of eddy currents on primary current and armature end magnetic flux on closing. Deviations from values in relay without eddy currents, in percent of final steady state value. Relay with magnetic leakage. Cases L87($\mu/10$) and S87($\mu/10$).

Still, the problem remains so complicated that it is hardly possible to deduce a useful exact solution, that is, to find from the operational expressions given in Part 3 numerically useful time functions. We therefore make use of the equivalent winding. We can do this with great confidence. In fact, for the simplified problem of Chapter 4, the equivalent winding gave us satisfactory results down to at least 5 ms, and neither the magnetic leakage nor the eddy currents in the yoke can be expected to bring about any basic change of our theory.

We have calculated, for the eight cases given in Table 1 of the preceding chapter, the equivalent winding parameters, and from these the armature end flux and the primary current on closing. For comparison the cases of eddy currents in the core only and of no eddy currents have also been worked out.

The results are collected in the four *Figs. 10 to 13.* The curves are drawn on a logarithmic time scale. In each case they present the deviation of the actual current or flux value from the corresponding value in a relay without eddy currents. The deviations are given in percent of the final steady state value.

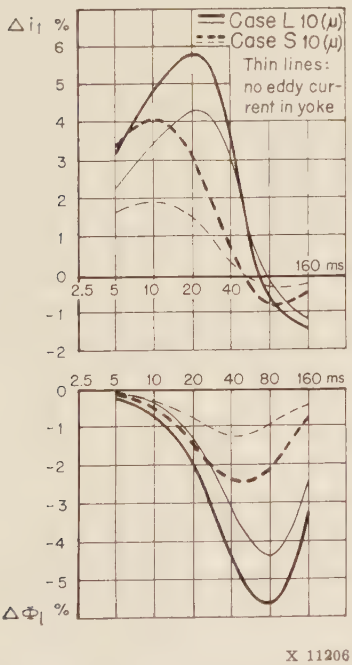


Fig. 12. Influence of eddy currents on primary current and armature end magnetic flux on closing. Deviations from values in relay without eddy currents, in percent of final steady state value. Relay with magnetic leakage. Cases L10(μ) and S10(μ).

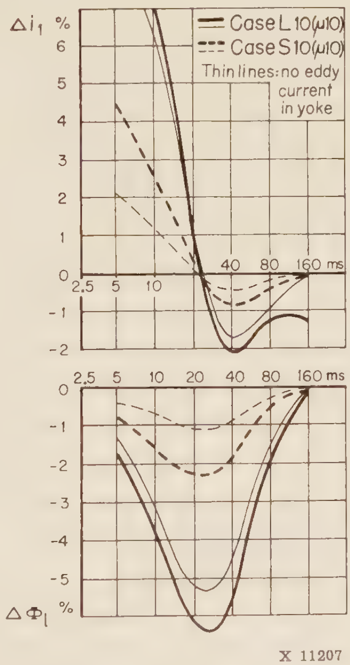


Fig. 13. Influence of eddy currents on primary current and armature end magnetic flux on closing. Deviations from values in relay without eddy currents, in percent of final steady state value. Relay with magnetic leakage. Cases L10(μ/10) and S10(μ/10).

Some general conclusions

The eddy current influence on current and magnetic flux in actual relays on closing is quite small. Therefore, it is not easy to compare our calculations with experimentally determined oscillograph records. From such a comparison it appears, however, that of the two series of permeabilities in Table 1, only the one with high values leads to values of current and flux which are close to the actual ones. By a suitable adjustment of these high values, one could no doubt obtain a very satisfactory agreement with the oscillograms.

In other words, during the closing operation the low initial permeability is of little importance. One should not find this astonishing. To be sure, for small time values the magnetic flux is small but its distribution across the core and yoke sections is strongly inhomogeneous. The flux density and consequently the permeability will therefore reach large values in those parts of the iron which effectively transmit the flux and where the permeability value is hence of importance. One can presume that the permeability value corresponding to the final steady state should yield good results down to fairly small time values.

In our study of the breaking operation in Part 3 the same argument was put forth, the flux on breaking being small but highly inhomogeneous for high time values.

We can therefore state that *both on closing and on breaking, a constant permeability value, in the neighbourhood of the* (*generally high*) *value corresponding to the D.C. steady state of the relay, will allow a good representation of the entire course of events.* In both cases the permeability values for small magnetic intensities are of little interest!

It appears that the facts discussed here have not always been duly understood. In the literature on telephone relays a homogeneous flux distribution is often assumed, whereas—on closing as well as on breaking—the magnetic flux in core and yoke is in general strongly inhomogeneous during the major part of the transient period. It should be observed, however, that when the flux passes over to the air gap, it almost immediately assumes a distribution which is very close to the one characterizing the static case. Hence, the inhomogeneity in the iron can be assumed to have only little influence on the pulling force on the armature. Actual force measurements support this conclusion. Its elucidation by further theoretical study would be of value.

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Principal symbols and formulæ

Symbols

t = time.
l = length of core and yoke (length of primary winding).
x = distance along core and yoke, counted from the heel piece end.
r = radius of core.
a = width of yoke.
c = thickness of yoke.

γ' = electric conductivity of core material.
γ'' = ,, ,, ,, yoke ,,
μ' = magnetic permeability of core material.
μ'' = ,, ,, ,, yoke ,, (In Part 3: γ_1, γ_2; μ_1, μ_2.)

N_1 = number of turns of primary winding.
S' = $s'l$ = reluctance of core.
S'' = $s''l$ = ,, ,, yoke. (In part 3: S_1, S_2.)
S = $S' + S'' = sl$ = reluctance of core + yoke.
Σ_0 = reluctance of heel piece end.
Σ_1 = ,, ,, armature + air gap.
S_λ = s_λ/l = total leakage reluctance.
R_1 = resistance of magnetizing (primary) circuit.
L_1 = inductance ,, ,, ,, ,,
R_2 = resistance of equivalent (secondary) winding.
L_2 = inductance ,, ,, ,, ,,
M = mutual inductance between primary and equivalent winding.
M_{11} = mutual inductance between primary winding and a fictitious winding of one turn at the armature end.
M_{21} = mutual inductance between equivalent winding and a fictitious winding of one turn at the armature end.

E_1 = constant emf in magnetizing circuit.
i_1 = $i_1(t)$ = current in primary winding.
$i_{1\infty}$ = E_1/R_1 = steady state value of primary current.
i_2 = $i_2(t)$ = current in equivalent winding (total circulating eddy current).
Φ = $\Phi(x, t)$ = total magnetic flux in core and yoke.
$\Phi_l(t) = \Phi(l, t)$ = magnetic flux at the armature end (in Parts 2 and 3: Φ_1).
Φ_∞ = $M_{11}\, E_1/R_1$ = steady state value of armature end flux.
φ_λ = $\varphi_\lambda(x, t)$ = magnetic leakage flux per unit core length.

Formulæ

Auxiliary quantities

$$a = \frac{\Sigma_0}{S_\lambda}; \quad b = \frac{\Sigma_1}{S_\lambda} \tag{29 a, b}$$

$$\eta = \sqrt{\frac{S}{S_\lambda}}; \quad \eta' = \sqrt{\frac{S'}{S_\lambda}}; \quad \eta'' = \sqrt{\frac{S''}{S_\lambda}} \tag{30 a, b, c}$$

$$g^2 = \eta^2 \frac{ab\dfrac{\sinh \eta}{\eta} + a\cosh \eta + b}{\eta^2 \dfrac{\sinh \eta}{\eta} + b(\cosh \eta - 1)} \tag{31 a}$$

$$\eta^2 \ll 1: \quad g^2 \approx \frac{a + b + ab + \dfrac{1}{2}\eta^2 a\left(1 + \dfrac{1}{3}b\right)}{1 + \dfrac{1}{2}b + \dfrac{1}{6}\eta^2\left(1 + \dfrac{1}{4}b\right)} \tag{31 b}$$

$$m_1 = 1 + a/2 - g^2/6 \tag{34 a}$$

$$n_1 = 1 + a/3 - g^2/12 \tag{34 b}$$

$$p_1 = 1 + a/4 - g^2/20 \tag{34 c}$$

$$q_1 = 1 + a - (g^2 - a^2)/3 - ag^2/4 + g^4/20 \tag{34 d}$$

$$r_1 = 1 + a - (g^2 - 2a^2)/6 - ag^2/8 \tag{34 e}$$

$$t_1 = 1 + a - (7g^2/4 - a^2)/5 - ag^2/8 + g^4/56 \tag{34 f}$$

$$\delta' = \frac{\gamma'/\pi}{\gamma'/\pi + \gamma'' c/2a}; \quad \delta'' = \frac{\gamma'' c/2a}{\gamma'/\pi + \gamma'' c/2a} \tag{40 a, b}$$

$$\beta = \frac{m_1 + \dfrac{1}{2}bn_1}{a + b + ab} \tag{59 b}$$

Equivalent winding parameters

$$R_2 = \frac{q_1}{m_1^2 l}\left(\frac{2\pi}{\gamma'}\,\delta'^2 + \frac{16a/3c}{\gamma''}\,\delta''^2\right) \tag{65}$$

$$L_2 = \frac{1}{12 m_1^2 S_\lambda}\left\{\left(1+\frac{\delta''}{3}\right)^2\left(1-\frac{m_1\eta^2}{g^2+\eta^2}\right)\left(3\beta r_1 - \frac{1}{2}t_1\right) + q_1\left(\frac{\delta'^2}{\eta'^2}+\frac{16}{15}\frac{\delta''^2}{\eta''^2}\right)\right\} \tag{70}$$

$$M = \frac{N_1}{2 m_1 S_\lambda}\left(1+\frac{\delta''}{3}\right)\frac{q_1}{g^2+\eta^2} \qquad \text{(75), } \textit{cf.} \text{ eq. (77)}$$

$$M_{21} = \frac{1}{2 m_1 S_\lambda}\left(1+\frac{\delta''}{3}\right)\left(1-\frac{m_1\eta^2}{g^2+\eta^2}\right)\left[\beta(1+a) - \frac{1}{2}n_1\right] \tag{79}$$

For $S_\lambda = \infty$, $\gamma'' = 0$ (no magnetic leakage, eddy currents in core only), see eqs. (82 a—f), (83 a—d).

Time constants (*cf. Part 2*)

$$T_1 = \frac{L_1}{R_1}; \quad T_2 = \frac{L_2}{R_2}; \quad T_{21} = T_2 - \frac{M_{21}}{M_{11}}\frac{M}{R_2}$$

$$\left.\begin{matrix}\tau_1\\ \tau_2\end{matrix}\right\} = \frac{T_1+T_2}{2} \pm \sqrt{\left(\frac{T_1+T_2}{2}\right)^2 - \sigma T_1 T_2}$$

where

$$\sigma = 1 - \frac{M^2}{L_1 L_2}$$

Primary current and armature end flux (*cf. Part 2*)

$$\frac{i_1}{i_{1\infty}} = 1 - \frac{(\tau_1 - T_2)e^{-\frac{t}{\tau_1}} - (\tau_2 - T_2)e^{-\frac{t}{\tau_2}}}{\tau_1 - \tau_2}; \quad i_{1\infty} = i_1(\infty) = \frac{E_1}{R_1}$$

$$\frac{\Phi_l}{\Phi_\infty} = 1 - \frac{(\tau_1 - T_{21})e^{-\frac{t}{\tau_1}} - (\tau_2 - T_{21})e^{-\frac{t}{\tau_2}}}{\tau_1 - \tau_2}; \quad \Phi_\infty = \Phi_l(\infty) = M_{11}\frac{E_1}{R_1}$$

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Frequency Components in the Output of a Harmonic Generator Driven by the Sum of Two Sine Waves

BY

GÖRAN EINARSSON*

	Page
Formulation of the Problem	140
Theory	141
Special Case: Small Amplitude Interference	145
Harmonic Generator for Odd Harmonics	147
Effect of Different Types of Interference	148
Bibliography	148

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This article derives a frequency analysis of the output of a harmonic generator which operates by emitting a constant amplitude pulse at each zero crossing of the input voltage. The input voltage is assumed to consist of the sum of two sine waves of arbitrary frequency plus a d.c. voltage. The result is given in the form of a definite integral.

For the case where one of the sine waves constitutes an interference of small amplitude, it is shown that the result can be expressed in the form of Bessel functions.

The problem is of especial interest in carrier systems where the requirements for purity of the carrier frequencies are very severe, and it is therefore important to know what effect interference at the input of the harmonic generator has on unwanted frequencies in the output spectrum.

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Formulation of the Problem

We assume that a harmonic generator, fed by a control voltage represented by the function F_s, delivers a pulse of unit height and duration 2δ for every zero crossing of the function F_s. The pulse train emitted constitutes the output function F_u (see *fig. 1*).

If F_s is a pure sine wave of angular frequency ω, then F_u contains only odd harmonics of ω. By varying the pulse width, we can obtain maximum amplitude for those harmonics in which we are interested. Since F_u is periodic, the amplitude of the ν:th harmonic can be shown by normal Fourier analysis to be

$$|B_\nu| = \frac{4}{\pi\nu} \left| \sin(\nu\omega\delta) \right| \qquad (1)$$

$$\nu_{\text{odd}}$$

An example of a harmonic generator working on the above principles is the magnetic harmonic generator described by PETERSON, MANLEY & WRATHALL[1], which employs an iron-cored inductor periodically driven to saturation.

If F_s is a sine wave plus a constant term (corresponding to a d.c. component), the zero crossings and therewith the pulses in F_u are shifted. The pulses thus become paired and in consequence F_u contains both odd and even harmonics. Since F_u remains periodic, the amplitude of the νth harmonic can readily be shown to be

$$|B_\nu| = \frac{4}{\pi\nu} \left| \sin(\nu \arccos Q) \cdot \sin(\nu\omega\delta) \right| \qquad (2)$$

where Q is the value of the constant term in F_s. ($Q<1$).

This means that both odd and even harmonics can be obtained by using a d.c. bias. By varying the pulse width and the amount of bias, we can vary the amplitude distribution of the harmonics which are of interest.

Next let F_s include another sine wave in addition to the constant term. This also produces shifts in the zero crossings and consequently shifts in the pulse positions in F_u. The output voltage will then contain not only harmonics of the two frequencies but also their combination frequencies. There is thus a possibility of using such a harmonic generator as a mixer.

In the general case when the ratio between the two input frequencies is not a rational number, F_u will not be periodic and normal Fourier analysis cannot be used. However, F_u can be regarded as a function of two variables and expanded as a double Fourier series, as shown in the sequel.

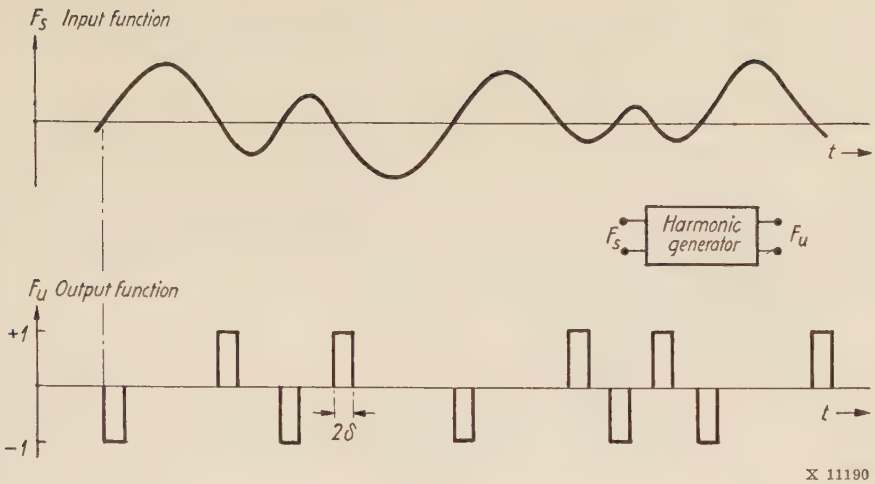

Fig. 1.

In practice one must be prepared for the control function F_s to contain interference of various kinds. This can for instance be power-frequency hum or, due to the control frequency itself having been generated in another harmonic generator, it can consist of residues of other harmonics incompletely removed by filtering.

In carrier systems a very high purity of the carriers is required, amongst other reasons because of the risk of crosstalk. It is therefore of great interest to be able to estimate the magnitude of the unwanted components at the output of a harmonic generator due to interference present in the input. For this purpose formulas are also derived for the output component amplitudes for the special case when the control function consists of a constant and two sine waves of which one is very small compared with the other.

Theory

Suppose

$$F_s = \cos x + k \cos y - Q$$

$$|Q| + |k| < 1$$

where

$$x = pt + \psi$$

$$y = qt + \varphi$$

With the harmonic generator described above, the output voltage depends only on the zeros of F_s. If we regard F_s as a function of two variables, it is periodic with respect to x

and y separately, with a period of 2π. If we similarly regard F_u as a function of two variables, we can develop a double Fourier series—see, for instance, HOBSON[2]

$$F_u = \frac{1}{2}A_{00} + \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}{}^{*} A_{\nu,\pm\mu}\cos(\nu x \pm \mu y) + \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}{}^{*} B_{\nu,\pm\mu}\sin(\nu x \pm \mu y) \tag{3}$$

where * signifies that the term for $\nu = \mu = 0$ is not included in the sum, and that the sum includes terms with both positive and negative signs except when one of the indices is zero, in which case only the terms with positive signs are taken.

The coefficients are obtained from the relations

$$A_{\nu,\pm\mu} = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_u(x,y)\cos(\nu x \pm \mu y)\,dxdy \tag{4}$$

$$B_{\nu,\pm\mu} = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_u(x,y)\sin(\nu x \pm \mu y)\,dxdy \tag{5}$$

Since both x and y contain t as a parameter, different values of t correspond to points along the straight line in the x, y plane given by the equations

$$x = pt + \psi$$

$$y = qt + \varphi$$

On the assumption that the pulse duration in F_u is constant, the form of $F_u(x, y)$ will depend not only on k and Q but also on p and q.

Figures 2 and *3* show the form of $F_u\ (x, y)$ for various values of the parameters concerned. The chain-dotted line is given by the equation

$$F_s = \cos x + k\cos y - Q = 0$$

The other bounding line is given by the condition, that all pulses have the same time duration 2δ. This means that the distance between the intersections of the two bounding lines and the straight line representing different values of t, and all lines parallell to it, is always equal to $2\delta\sqrt{p^2 + q^2}$. The projection of this distance on the x- and y-axis is then $2\delta p$ and $2\delta q$, as it should be.

It will be seen from *Fig. 3* that the two limiting lines can intersect if $qk > p$. When this occurs the interval between two zeros of F_s can be less than the pulse duration 2δ, so that the second pulse could occur before the previous pulse had finished. On the assumption that simultaneous positive and negative pulses cancel to give $F_u = 0$, then F_u takes the form shown in *Fig. 3*. If the actual operation of the harmonic generator is different from this

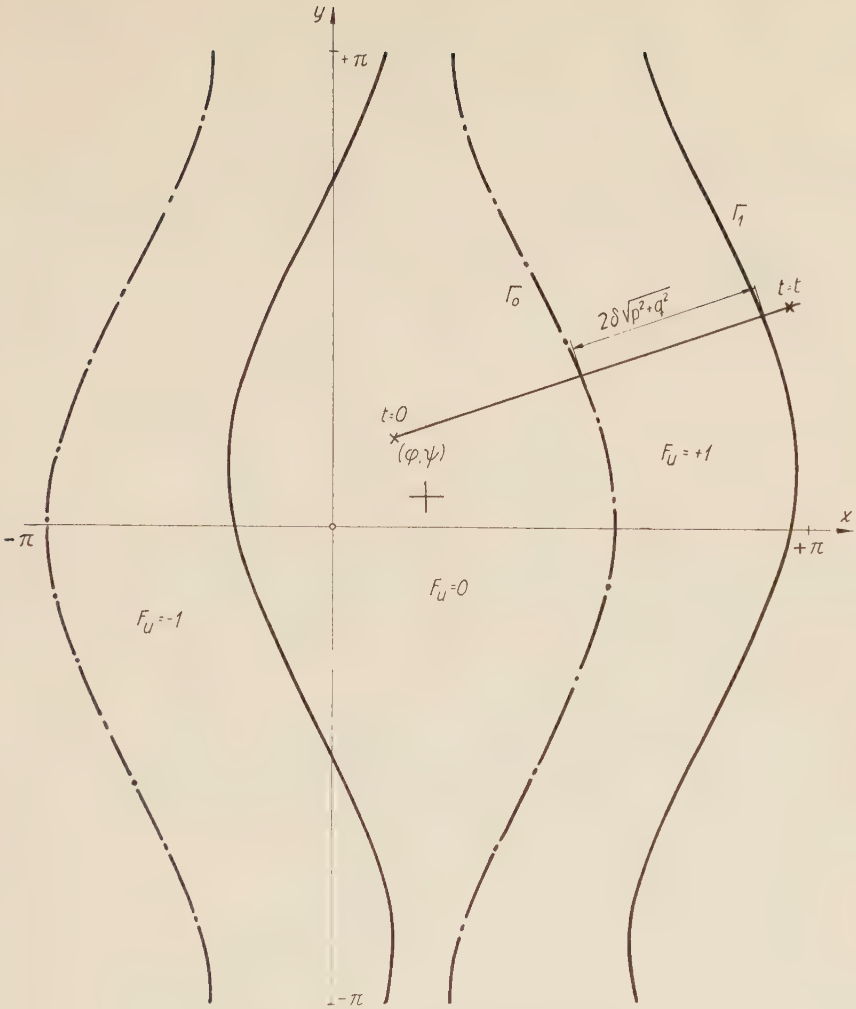

Fig. 2. $k = 0.5 \quad Q = 0.2 \quad p/q = 3.$

assumption, then the form of F_u must be correspondingly modified from that shown in *Fig. 3.*

It is assumed that the pulse width is not so large that the strips in *Figs. 2* and *3* merge together.

The equations for the two bounding lines, expressed as $x = f(y)$, are

$$\Gamma_0(y) = \arccos(Q - k\cos y) \tag{6}$$

$$\Gamma_1(y) = \arccos[Q - k\cos(y - 2\delta q)] + 2\delta p \tag{7}$$

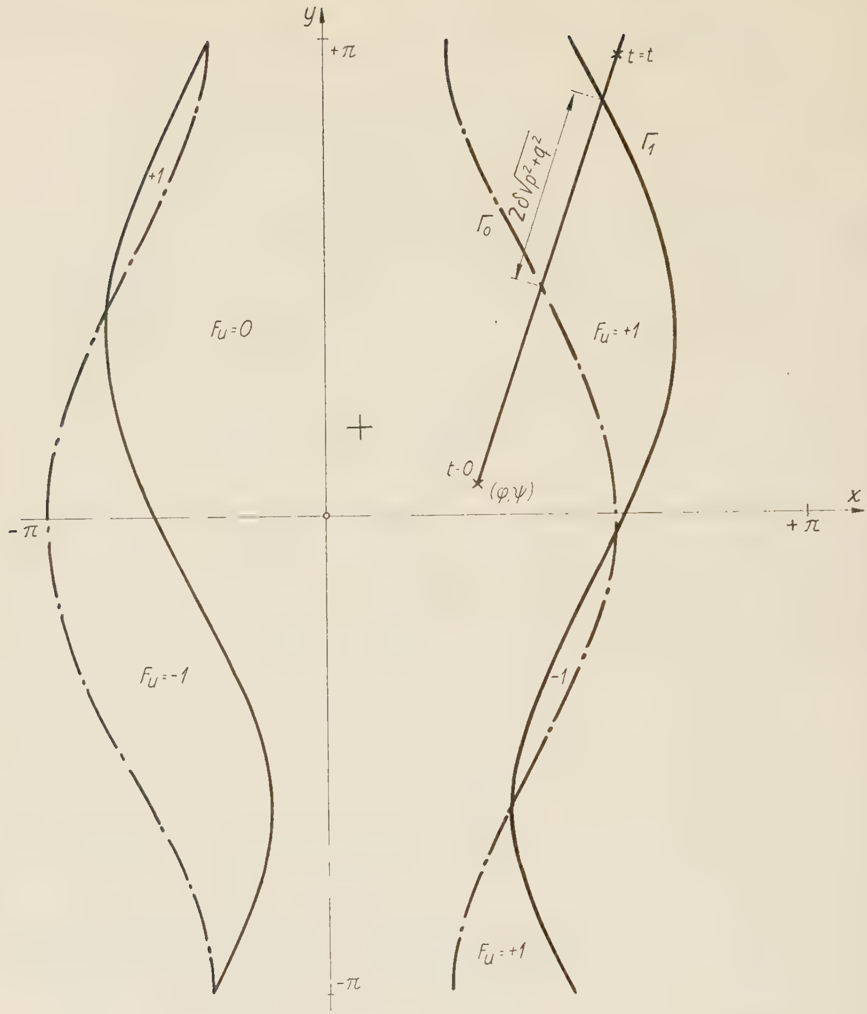

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Fig. 3. $k = 0.5 \quad Q = 0.2 \quad p/q = {}^1/_3$.

Since F_u is periodic in x and y, we can determine the coefficients in the double Fourier series, thus obtaining an expression valid over the whole of the x, y plane and in particular for the values along the line representing various values of t.

To simplify the calculation we introduce a new co-ordinate system (x', y') with axes parallel to the old ones but with origin at the point $(\delta p, \delta q)$. $F_u(x', y')$ thereby becomes an odd function and from (4) we find that

$$A_{\nu, \pm \mu} = 0$$

The equations of the bounding lines in the new co-ordinate system are

$$\Gamma_0(y') = \arc\cos[Q - k\cos(y' + \delta q)] - \delta p \tag{8}$$

$$\Gamma_1(y') = \arc\cos[Q - k\cos(y' - \delta q)] + \delta p \tag{9}$$

By integrating only over the regions where $F_u = \pm 1$, we obtain from equation (5):

$$B_{\nu,\pm\mu} = \frac{1}{\pi^2}\int_{-\pi}^{+\pi}\left[\int_{\Gamma_0(y')}^{\Gamma_1(y')}\sin\nu x'dx'\right]\cos\mu y'dy' \pm \frac{1}{\pi^2}\int_{-\pi}^{+\pi}\left[\int_{\Gamma_0(y')}^{\Gamma_1(y')}\cos\nu x'dx'\right]\sin\mu y'dy' \tag{10}$$

where $\Gamma_1(y')$ and $\Gamma_0(y')$ are given by (8) and (9). Carrying out the internal integrations in (10) we have:

$$B_{\nu,\pm\mu} = \frac{1}{\nu\pi^2}\int_{-\pi}^{+\pi}[\cos\nu\Gamma_0(y') - \cos\nu\Gamma_1(y')]\cos\mu y'dy' \pm$$

$$\pm\frac{1}{\nu\pi^2}\int_{-\pi}^{+\pi}[\sin\nu\Gamma_1(y') - \sin\nu\Gamma_0(y')]\sin\mu y'dy' \tag{11}$$

In the above treatment we have assumed that $|Q| + |k| < 1$. If we let $(1 - |k|) < |Q| < (1 + |k|)$ the integration area assumes a different form from that shown in *Figs. 2 and 3.* The bounding lines then become ellipses and the same treatment as above can be used.

Special Case: Small Amplitude Interference

If we assume that one of the frequencies constituting F_s consists of an interference of small amplitude, then a solution in the form of Bessel functions can be obtained.

We assume $k \ll 1$ and expand arc cos $(Q + Z)$ by means of a Taylor series about the point Q.

$$\arc\cos(Q + Z) = \arc\cos Q - \frac{Z}{\sqrt{1 - Q^2}} \pm \ldots. \tag{12}$$

where $\qquad Z = -k\cos y$

Taking only the first two terms in (12) we can write (8) and (9)

$$\Gamma_0 \simeq \arc\cos Q + \frac{k}{\sqrt{1 - Q^2}}\cos(y' + \delta q) - \delta p \tag{13}$$

$$\Gamma_1 \simeq \arc\cos Q + \frac{k}{\sqrt{1 - Q^2}}\cos(y' - \delta q) + \delta p \tag{14}$$

$k \ll 1$

Expression (11) can be written as

$$B_{\nu,\pm\mu} = Im\frac{1}{i\nu\pi^2}\int_{-\pi}^{+\pi}(e^{i\nu\Gamma_1} - e^{i\nu\Gamma_0})e^{\pm i\mu y'}dy' \tag{15}$$

Substituting (13) and (14) in (15) we obtain after a change of integration variable

$$B_{\nu, \pm\mu} \simeq Im \frac{1}{i\nu\pi^2} e^{i\nu \arccos Q} \left[e^{i(\nu\delta p \pm \mu\delta q)} \int_{-\pi-\delta q}^{+\pi-\delta q} e^{i\left[\nu \frac{k}{\sqrt{1-Q^2}} \cos y \pm \mu y\right]} dy - \right.$$

$$\left. - e^{-i(\nu\delta p \pm \mu\delta q)} \int_{-\pi+\delta q}^{+\pi+\delta q} e^{i\left[\nu \frac{k}{\sqrt{1-Q^2}} \cos y \pm \mu y\right]} dy \right]$$

Using the formulas

$$\int_{\alpha}^{\alpha+2\pi} e^{i(z\cos y + \mu y)} dy = 2\pi e^{\frac{i\mu\pi}{2}} J_{\mu}(Z) \tag{16}$$

$$\mu = 0, \ \pm 1, \ \pm 2, \ldots .$$

$$J_{-\mu}(Z) = (-1)^{\mu} J_{\mu}(Z) \tag{17}$$

we obtain for $k \ll 1$

$$B_{\nu, \pm\mu} \simeq \frac{4}{\nu\pi} \sin \delta(\nu p \pm \mu q) \cdot \sin\left(\nu \arccos Q + \frac{\mu\pi}{2}\right) \cdot J_{\mu}\left(\frac{\nu k}{\sqrt{1-Q^2}}\right) \tag{18}$$

For small values of the argument we can expand $J_{\mu}(Z)$ as a series:

$$J_{\mu}(Z) = \frac{\left(\frac{1}{2}Z\right)^{\mu}}{\underline{|0}\,\underline{|\mu}} - \frac{\left(\frac{1}{2}Z\right)^{\mu+2}}{\underline{|1}\,\underline{|\mu+1}} \pm \ldots . . \tag{19}$$

Taking the first terms of (19), we obtain finally

$$B_{\nu, \pm\mu} \simeq \frac{4}{\nu\pi} \sin \delta(\nu p \pm \mu q) \cdot \sin\left(\nu \arccos Q + \frac{\mu\pi}{2}\right) \frac{\left(\frac{1}{2} \frac{\nu k}{\sqrt{1-Q^2}}\right)^{\mu}}{\underline{|\mu}} \tag{20}$$

for $\nu k \ll 1$

The desired harmonics, *i.e.* the multiples of the angular frequency p, are seen from (20) to have the amplitudes

$$B_{\nu, 0} \simeq \frac{4}{\nu\pi} \sin \delta\nu p \cdot \sin(\nu \arccos Q) \tag{21}$$

which agrees with expression (2).

The largest amplitudes for interference products in the output spectrum are obtained for $\mu = 1$: that is, the angular frequencies of $\nu p \pm q$ are the most troublesome.

The ratio between a desired harmonic and its nearest interference product is

$$\left|\frac{B_{\nu,\pm 1}}{B_{\nu,0}}\right| \simeq \frac{\sin\delta(\nu p \pm q)\cdot\cos(\nu \arccos Q)}{\sin\delta\nu p} \frac{\nu k}{\sin(\nu\arccos Q)\, 2\sqrt{1-Q^2}} \tag{22}$$

It will be seen from the above that an interfering frequency of small amplitude in the control function gives rise to combination frequencies in the output spectrum in the neighbourhood of the wanted harmonics. The important fact is that the suppression of unwanted frequencies in the output spectrum gets worse as the order of these harmonics increases. Harmonic generators with a high multiplication ratio therefore require a very high suppression of, for instance, power mains hum voltage if the extracted harmonics are not to be modulated by the hum frequency. This applies to harmonic generators where the control function includes a bias, and both odd and even harmonics are produced. For harmonic generators without bias yielding odd harmonics only the conditions are more favourable, as will be seen in the next section.

Harmonic Generator for Odd Harmonics

If we assume $k \ll 1$ and $Q = 0$, then expression (20) yields directly

$$\begin{aligned} &B_{\nu,\pm\mu} = 0 && \text{for } (\nu+\mu) \text{ even} \\ &B_{\nu,\pm\mu} \simeq (-1)^{\frac{\nu+\mu-1}{2}} \frac{4}{\nu\pi} \sin\delta(\nu p \pm \mu q) \frac{\left(\frac{1}{2}\nu k\right)^{\mu}}{\underline{|\mu}} && \text{for } (\nu+\mu) \text{ odd} \end{aligned} \tag{23}$$

Here also the largest interference products are found for $\mu = 1$, corresponding to angular frequencies $(\nu \pm 1)\, p \pm q$.

ν_{odd}

A significant difference from the case of $Q \neq 0$ is that the interference products are not grouped around the wanted odd harmonics, but instead are found around the positions of the even harmonics.

If $q \ll \nu p$, the ratio between interference and harmonic is

$$\left|\frac{B_{\nu\pm 1,\pm 1}}{B_{\nu,0}}\right| \simeq \frac{\nu k}{2} \quad \text{for } \nu \text{ odd} \tag{24}$$

In this type of harmonic generator low frequency interference in the control function is not dangerous, since the corresponding interference in the output spectrum falls near adja-

cent even harmonics and can therefore readily be removed by filters. An interference frequency near the fundamental frequency, however, gives rise to interference products near the odd harmonics.

Formula (24) has been given by PETERSON.[1]

Examples of the Effects of Different Types of Interference

1. The interference consists of a d.c. voltage

$k = 0, \; Q \ll 1$

In this case both odd and even harmonics are obtained with an amplitude ratio

$$\left|\frac{B_{\nu \pm 1}}{B_\nu}\right| \simeq \nu Q \quad \text{for } \nu \text{ odd and } \nu Q \ll 1$$

This result is obtained directly from expression (2), or by putting $q = 0$ in expression (18) and summing over μ.

2. Interference consists of second harmonic

$Q = 0, \quad k \ll 1, \quad q = 2p.$

In this case both odd and even harmonics are obtained with an amplitude ratio.

$$\left|\frac{B_{\nu \pm 1}}{B_\nu}\right| \simeq \nu k \quad \text{for } \nu \text{ odd and } \nu k \ll 1$$

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CONTENTS

		Page
Extension of Telephone Plant with Regard to the Value of Subscribers' Time. III. The Reference Equivalent within a National Network	Y Rapp	151
On the Dependence between the Two Stages in a Link System	A Elldin	183
Alternative Routing in a Two-Stage Link System. Congestion Theory and Simulated Traffic Studies	B Wallström	261

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STOCKHOLM 32

Extension of Telephone Plant with Regard to the Value of Subscribers' Time

III. The Reference Equivalent within a National Network

BY

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LME 8077

In a previous paper, *Economic Optimum for Conductor Diameter and Number of Switches on Telephone Circuits* (Ericsson Technics No. 1, 1961), the principle was expounded of how, from an evaluation of the inconvenience to subscribers due to inadequate accessibility and intelligibility, means can be created to provide guidance in deciding the number of switches on traffic routes and the reference equivalent of telephone circuits.

The present paper will deal, in greater detail than was possible in the previous paper, with the question of the value of the reference equivalent in a national network. This will be done by using the results obtained in respect to the increase of conversation time at different transmission levels and at normal room noise level, as reported in a separate contribution to the next issue, *Estimation of the Increase in Conversation Time as Function of the Overall Reference Equivalent of a Telephone Circuit,* by H. Hansson and G. Lind. It should perhaps be emphasized that the present paper has not, and cannot have, as its aim to present recommendations concerning the value of reference equivalent for national circuits. Its purpose is solely to throw light on the question from the economic aspect, with the value of the subscribers' time as criterion. The questions discussed in Parts I and II were approached on essentially the same lines, which is the reason for adopting the common title, *Extension of Telephone Plant with Regard to the Value of Subscribers' Time.*

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Contents

Page

CHAPTER 1 Object of the Paper and Survey of the Problem 153

CHAPTER 2 General Principles for Calculation of Conductor Diameters and Reference Equivalent within a National Network on the Basis of the Value of Subscribers' Time 155

CHAPTER 3 The Reference Equivalent of a National Circuit between Two Identical Networks 158

CHAPTER 4 The Reference Equivalent when Repeaters are Used on the Periphery of the Network 165

CHAPTER 5 Principles for Division of a Given Overall Reference Equivalent between Sending and Receiving within a National Network 168

ANNEX 1 The Reference Equivalent under the Assumption that the Prolongation Factor for the Peripheral Subscribers is Used as the Basis for Calculating the Time Losses of All Subscribers 175

ANNEX 2 The Reference Equivalent on the Assumption that the Subscriber Distribution is Rectangular 177

ANNEX 3 Simplified Approximate Method for Determination of Mean Prolongation Factor, ϑ_m 179

ACKNOWLEDGEMENTS 182

BIBLIOGRAPHY 182

CHAPTER 1

Object of the Paper and Survey of the Problem

The attempt made in this paper to assist in determining the reference equivalent for a national network rests—as did the previous paper under the same main title in Ericsson Technics No. 1, 1961—on the supposition that the inconvenience to subscribers due to unsatisfactory transmission is measured by the prolongation of conversation time which may be anticipated when the overall reference equivalent on a telephone circuit exceeds a given level, and is evaluated in monetary terms by means of a factor G, which may be said to represent the value of the subscribers' time.

Traditional measures of intelligibility or records of the subjective opinion of subscribers on the quality of circuits at different levels of transmission are not used in this presentation. The reason is, not that such tests are considered imperfect tools, but quite simply that the author hopes it may be of some interest to treat, from a common *economic* aspect, a group of questions within telephony which—for instance traffic losses and transmission losses—may at first sight seem incommensurable.

This paper does not claim to present a definite answer to the question of the most suitable level of reference equivalent within a national network; but it does claim to add to the arguments in favour of lowering the present recommendations one further argument which rests on economic grounds.

In chapter 2 the principles already discussed in Ericsson Technics No. 1, 1961, pp. 11—14 and 41—42, will be extended to apply to an arbitrary network configuration. For the reader's convenience the portions necessary for an understanding of the context will be repeated. Chapter 2 forms the basis for the subsequent treatment. Its chief interest, however, is theoretical, since so general a treatment would lead to extremely extensive numerical calculations.

Furthermore the problem as formulated in this chapter lies to some extent outside the scope of this paper, the object of which is not to find the scheme which, for a given configuration and traffic distribution, yields the minimum overall costs, but is in fact far more modest, namely, to throw light on the question from the economic aspect and to attempt to give certain probable limits for the reference equivalent, based on the cost of the cable plant, the value of subscribers' time, and the increase of conversation time.

In formulating the object of the study in this way, we shall leave out of account the question—of interest at least from the econometric point of view—of whether the reference equivalent within a national network should be differentiated, i.e. be given different values for different directions of traffic, depending, for example, on the part of the reference equivalent which is independent of the conductor diameter or on the quantity of traffic.

Chapter 2 may be regarded as a preliminary draft for the treatment of this more general question. With the aforementioned objective, however, it is sufficient, at least to start with,

as is done in chapter 3, to study the question from the angle of two identical networks joined by a trunk cable. The calculations presented in chapter 3, which are based on the use of a single conductor diameter in the network, show that, whereas the reference equivalent should be fairly low for 99 per cent of all calls, for the remaining 1 per cent it may be fairly high, especially in networks with few subscribers on the periphery.

This gives cause, in chapter 4, to take up the question of the use of repeaters for subscribers on the periphery from the economic aspect.

In chapter 5 the question of the maximum permissible reference equivalent is considered as having been decided, and it is shown how an already fixed reference equivalent should be divided between sending and receiving so as to minimize the sum of the cable plant costs.

The numerical examples which are given seem to lend support, based on economic reasoning, to the view that the maximum reference equivalent in a national network should be appreciably reduced. Viewed from this angle, it seems that the reference equivalent for a national circuit should not exceed 32—34 db, at least for 99 per cent of all calls.

Support is also given to the view that the division of the maximum permissible reference equivalent between sending and receiving should not be fixed once and for all, but be adapted to the special network configurations within different areas.

Finally, the opinion is corroborated that an improvement in the properties of telephone sets should not be entirely utilized to provide the subscribers with better transmission conditions nor simply to reduce the cost of the cable plant, but that such improvements shall be suitably divided between subscribers and cable plant.

In Annexes 1—3 it is shown how the value of the reference equivalent can be determined by simplified assumptions (Annexes 1—2) and by approximations (Annex 3). A study of these annexes is especially recommended to the reader who wishes to acquire a preliminary survey of the problem.

CHAPTER 2

General Principles for Calculation of Conductor Diameters and Reference Equivalent within a National Network on the Basis of the Value of Subscribers' Time

The purpose of this chapter is to show in principle how, for a given network configuration in which the traffic in all directions is known, and for a given invariable technical standard, the conductor diameters in the cables shall be determined so as to minimize the sum of the cable plant costs and the value of the subscribers' losses of time.

With the assumption of an invariable technical standard it follows, for example, that the transmission properties of the telephone sets, the attenuation in switching equipments and the reference equivalent on 4-wire circuits may be regarded as known. The room noise level and circuit noise level are also regarded as constant.

The variable magnitudes on a circuit between two subscribers are thus solely the conductor diameter in subscribers' and junction cables and the degree of loading of the latter. To simplify the picture, however, it is assumed that the junction cable attenuation as well may be regarded as a constant and thus be included in the sum of the constant attenuations referred to in the preceding paragraph. Finally, it is assumed that the conductor diameters may be continuously varied. No account is taken of technical constraints, such as that conductor diameters below a given limit are not employed or that the resistance in a loop may not exceed a given value. The reference equivalent for every circuit is thereby given as a function of the conductor diameters in the two networks concerned, and the problem is to determine the conductor diameters in all networks so as to minimize the sum of the cable plant costs and of the subscribers' losses of time due to prolongation of the conversation time. If one succeeds in carrying out such a calculation, it is obviously an easy matter to determine the reference equivalent in all conceivable directions. The expression to be minimized is

$$\sum_{\nu} c_\nu \cdot l_\nu \cdot z_\nu^2 \cdot N_\nu + 2G \cdot \sum_{\mu,\,\nu} T_{\mu\nu} \cdot (\vartheta_{\mu\nu} - 1) \tag{2.1}$$

where

c_ν = a constant, kr/mm² km

l_ν = the mean length of line in the network, km

z_ν = the conductor diameter in the network, mm

N_ν = the number of subscribers in the network

G = a constant representing the value of the subscribers' time, kr

$T_{\mu\nu}$ = the expected total conversation time in hrs/day under ideal transmission conditions in the direction $\mu \to \nu$

$\vartheta_{\mu\nu}$ = the mean prolongation factor for calls in the direction $\mu \to \nu$.

Tests conducted by H. Hansson and statistically analyzed by G. Lind[1] show that conversation time is likely to be prolonged as soon as the reference equivalent on a circuit exceeds a given value $R_* = 30$ db and that the prolongation factor can be described by an exponentially growing function of the overall reference equivalent, R db, in accordance with the following formula for the prolongation factor:

$$\begin{aligned} \vartheta &= e^{\gamma (R - R_*)^2} \qquad & R > R_* \\ \vartheta &= 1 & R \leq R_* \end{aligned} \tag{2.2}$$

where γ and R_* are constants with the estimated values $\gamma = 7.34 \cdot 10^{-4}$ and $R_* = 30$ db. The process of this function is shown in *Fig. 2.1.*

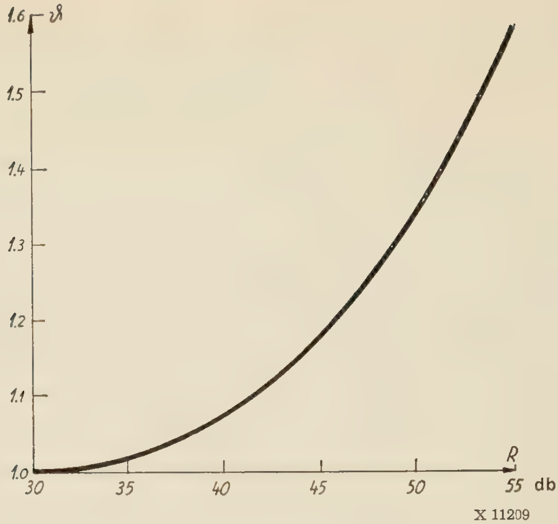

Fig. 2.1. The prolongation factor $\vartheta = e^{7.34\,(R-30)^2 \cdot 10^{-4}}$ as function of the overall reference equivalent R according to investigations made by H. Hansson and G. Lind. Room noise level 50 db relative to $2 \cdot 10^{-4}$ dynes/cm^2.

The prolongation factor on a circuit between two subscribers can thus be established as soon as the reference equivalent for the circuit is known.

The reference equivalent, $R_{yx}^{(\mu\nu)}$, for a circuit between a subscriber in a network μ at a distance y km from exchange no. μ to another subscriber in a network ν at distance x km from exchange no. ν is

$$R_{yx}^{(\mu\nu)} = (\alpha_\mu + \alpha_{\mu m}) \cdot y + \alpha_\nu \cdot x + R_{0\mu\nu} \tag{2.3}$$

where

$\alpha = \dfrac{c_1}{z}$ = a.c. attenuation, db/km

$\alpha_m = \dfrac{c_2}{z^2}$ = transmitter feed attenuation, db/km

$c_1,\ c_2$ = constants

z = conductor diameter, mm

$R_{0\mu\nu}$ = reference equivalent independent of the conductor diameter in the networks

One notices that generally, whereas $R_{0\mu\nu} = R_{0\nu\mu}$, $R_{yx}^{(\mu\nu)} \neq R_{xy}^{(\nu\mu)}$.

It remains to establish the mean prolongation of conversation time, $\vartheta_{\mu\nu} - 1$. For this a knowledge is required of the frequency functions $f_i(u)$, $(i = 1, 2 \ldots)$, for the geographical

distribution of the subscribers, and of the prolongation factor, ϑ_{yx}, on a circuit between two subscribers at distances y and x from their respective exchanges. Then

$$\vartheta_{\mu\nu} - 1 = \iint_{\Omega_{\mu\nu}} f_\mu(y) \cdot f_\nu(x) \cdot (\vartheta_{yx} - 1) \cdot dy \cdot dx \tag{2.4}$$

with the integration domain

$$\Omega_{\mu\nu}: \begin{cases} (\alpha_\mu + \alpha_{\mu m}) \cdot y + \alpha_\nu \cdot x + R_{0\mu\nu} \geq R_* \\ y \leq L_\mu \qquad x \leq L_\nu \end{cases}$$

Here L_μ, L_ν are the maximum lengths of line in the respective networks.

Numerical calculations with the aid of eq. 2.4 can be carried out by representing the subscriber distribution by a discrete distribution.

Records of the geographical distribution of subscribers in some twenty networks of different characters in different parts of the world, which have been statistically analyzed by G. LIND[2], show that the material studied can be described by a compound Poisson distribution with two parameters, a and g, according to the formula

$$p(x) = \left(\frac{g}{g+1}\right)^a \cdot \binom{a-1+x}{x} \cdot \left(\frac{1}{g+1}\right)^x \tag{2.5}$$

In this discrete distribution $x = 0$ represents the 0—0.5 km class, $x = 1$ the 0.5—1 km class, etc.

A survey has now been given of the means required for calculation of the costs of the cable plants and of the subscribers' time according to eq. 2.1 under different assumptions concerning the conductor diameters in the various cables. It is thereby also possible, in principle, to establish the conductor diameters, z_1, z_2 . . ., which give the least value for this expression. A necessary condition for this is obviously that the partial derivatives in respect of z_1, z_2 . . . shall be equal to zero. To illustrate these conditions in a simple way, we assume that the entire network can be represented by three networks 1, 2 and 3, and that the prolongation factor is unity for the traffic within the networks, which gives

$$\left.\begin{aligned} c_1 \cdot l_1 \cdot N_1 &= -G \cdot \frac{1}{z_1} \cdot \frac{\partial}{\partial z_1} (T_{12} \cdot \vartheta_{12} + T_{21} \cdot \vartheta_{21} + T_{13} \cdot \vartheta_{13} + T_{31} \cdot \vartheta_{31}) \\ c_2 \cdot l_2 \cdot N_2 &= -G \cdot \frac{1}{z_2} \cdot \frac{\partial}{\partial z_2} (T_{12} \cdot \vartheta_{12} + T_{21} \cdot \vartheta_{21} + T_{23} \cdot \vartheta_{23} + T_{32} \cdot \vartheta_{32}) \\ c_3 \cdot l_3 \cdot N_3 &= -G \cdot \frac{1}{z_3} \cdot \frac{\partial}{\partial z_3} (T_{13} \cdot \vartheta_{13} + T_{31} \cdot \vartheta_{31} + T_{23} \cdot \vartheta_{23} + T_{32} \cdot \vartheta_{32}) \end{aligned}\right\} \tag{2.6}$$

It is already evident that this manner of approaching the problem leads to extremely extensive numerical calculations, even if the national network is represented by a model consisting of only a few exchanges.

The reference equivalent for a national network must for natural reasons be established for a fairly long period, perhaps 10—20 years. In order to be able to calculate, in the manner indicated, an economically suitable value for this reference equivalent, one must clearly make a long-term prognosis both of the traffic distribution and of the network configuration. This immediately introduces a considerable element of uncertainty into the calculations. Furthermore the objection may perhaps be advanced against the actual principle entailed in the calculations, that the reference equivalents of different circuits will differ fairly considerably, depending on the traffic and on the general structure of the network, and that it is desirable to have a uniform and simple recommendation concerning the value of the reference equivalent.

Finally, as has already been expressly stated, a recommendation concerning the value of the reference equivalent must not be based exclusively on economic calculations; account must also be taken of tests based on the subjective opinions of subscribers.

All these reasons suggest that it is worth while to investigate whether one cannot, with simpler models, arrive at results which can at least set an upper limit to the reference equivalent from the economic aspect.

This will be done in the following chapters.

CHAPTER 3

The Reference Equivalent of a National Circuit between Two Identical Networks

We assume that the entire national network can be represented by two exchanges with two identical cable plants.

The condition in eq. 2.6 for a cost minimum is then reduced to a single equation, which may be suitably written

$$\frac{c \cdot l_m}{GT_0} = -\frac{1}{z} \cdot \frac{\partial \vartheta_m}{\partial z} \tag{3.1}$$

where

T_0 = expected total duration of calls in hours initiated per subscriber per day under ideal transmission conditions for trunk calls

G — the quality of service factor, representing the capitalized value of 1 hour per day of the subscriber's time, in kronor

c = a constant, kr/mm^2 km

l_m = mean length of line in the networks, km

z = conductor diameter in the networks, mm

For the sake of convenience the left-hand side of eq. 3.1 will be denoted by $\xi = \frac{c \cdot l_m}{GT_0}$. To make the calculation, a knowledge is also required of the prolongation of conversation time, which is obtained from*

$$\vartheta_m - 1 = \iint_{\Omega} f(y) \cdot f(x) \cdot (\vartheta_{yx} - 1) \cdot dy \cdot dx \tag{3.2}$$

with the integration domain

$$\Omega: \begin{cases} (\alpha + \alpha_m) \cdot y + \alpha \cdot x + R_0 \geq R_* \\ y \leq L \qquad\qquad x \leq L \end{cases}$$

where

L = longest subscriber's line in the networks, km.

Furthermore (cf. eq. 2.2 and 2.3)

$$\vartheta_{yx} = e^{\gamma(R_{yx} - R_*)^2} \tag{3.3}$$

$$R_{yx} = (\alpha + \alpha_m) \cdot y + \alpha \cdot x + R_0 \tag{3.4}$$

Our aim is to attempt to establish an upper limit for the reference equivalent. For this purpose we choose a network of large area, L = 5 km, and a relatively high value for the part of the reference equivalent, R_0 = 20 db, which is independent of the conductor diameter. The geographical distribution of the subscribers is shown in *Table 3.1*, which is taken from G. LIND's paper[2] in a later issue.

Table 3.1. Geographical distribution of subscribers according to investigations by G. Lind.
Longest subscriber's line in the network L = 5 km

Distance from exchange km	Distributions with		
	extremely short tail a	normal tail b	extremely long tail c
0 —0.5	0.2192	0.1554	0.1631
0.5—1	0.3087	0.2114	0.1443
1 —1.5	0.2391	0.1958	0.1277
1.5—2	0.1347	0.1533	0.1130
2 —2.5	0.0617	0.1089	0.1000
2.5—3	0.0243	0.0727	0.0885
3 —3.5	0.0086	0.0464	0.0783
3.5—4	0.0028	0.0287	0.0693
4 —4.5	0.0008	0.0172	0.0613
4.5—5	0.0002	0.0103	0.0543

* In Annex 3 is shown how the double integral (3.2) can often be replaced with good approximation by a single integral.

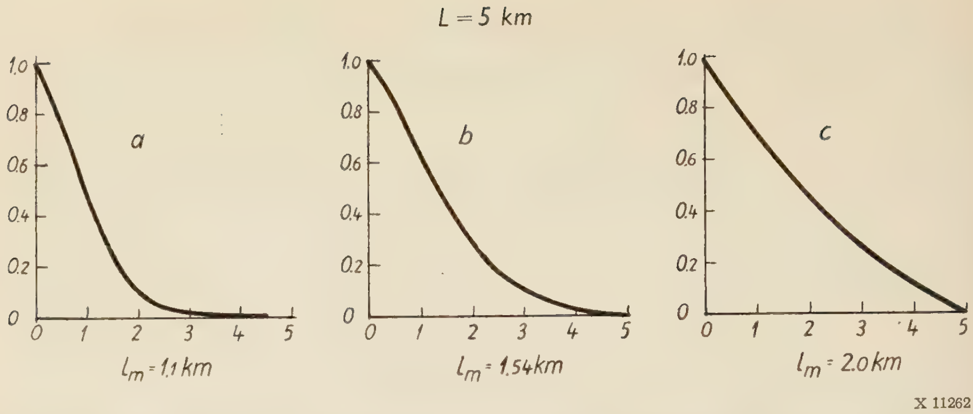

Fig. 3.1. Distribution functions of the geographical distribution of subscribers, according to G. Lind

a = cases with extremely short tail
b = normal cases
c = cases with extremely long tail
l_m = mean length of line, km

The distribution functions corresponding to the values in *Table 3.1* are shown in *Fig. 3.1.*

The a.c. attenuation and the transmitter feed attenuation are calculated from, respectively,

$$\alpha = \frac{0.58}{z} \text{ db/km} \qquad \alpha_m = \frac{0.19}{z^2} \text{ db/km} \tag{3.5}$$

where z, as before, is the conductor diameter in mm.

The necessary data for the numerical calculations have now been accounted for. The calculations are carried out by choosing a series of values for the conductor diameter, z, and calculating ϑ_m according to eq. 3.2, using the classification as in *Table 3.1.* $-\frac{1}{z}\cdot\frac{\partial\vartheta_m}{\partial z}$ is thereafter calculated with known interpolation formulae, using five mutually related values of ϑ_m and z.

The final step is to calculate

$$R_{\max} = (2\alpha + \alpha_m)\,L + R_0 \tag{3.6}$$

after which the desired relation between the factor ξ (see eq. 3.1) and $R_{\max}$ will have been determined. *Fig. 3.2* shows the result of the numerical calculations carried out on these lines.

The constant c in the factor ξ has been estimated, after a special investigation, as max. 500 kr/mm² km.

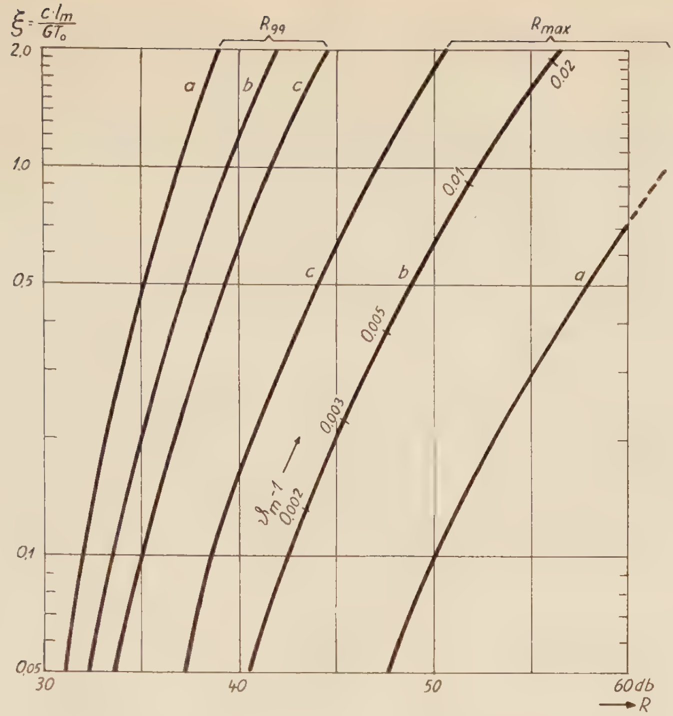

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Fig. 3.2. The relation between the factor $\xi = \dfrac{c \cdot l_m}{GT_0}$ and the reference equivalent R.

R_{max} = maximum reference equivalent, cf. eq. 3.6
R_{99} = upper limit of reference equivalent for 99 per cent of calls
L = 5 km $\quad R_0 = 20$ db
l_m = mean length of line, km
z = conductor diameter in mm
$\vartheta_{yx} = e^{7.34 \cdot 10^{-4} (R_{yx} - 30)^2}$
$R_{yx} = (\alpha + \alpha_m) \cdot y + \alpha \cdot x + 20$
a = cases with extremely short tail, $l_m = 1.1$ km
b = normal cases, $l_m = 1.54$ km
c = cases with extremely long tail, $l_m = 2.0$ km
$\vartheta_m - 1$ = mean prolongation of conversation time for normal cases (b)

The value of the subscribers' time on trunk calls, which should be greater than the cost of a call over the longest circuit within the national network, is estimated at $g = 80$ kr/hr,* which should be a reasonable lower limit.

* The corresponding value of G is $G = \dfrac{365 \cdot g}{r}$ where r = interest factor. With $r = 0.08$ one obtains $G = 365{,}000$.

See *Extension of Telephone Plant with Regard to the Value of Subscribers' Time. I. Economic Optimum for Conductor Diameter and Number of Switches on Telephone Circuits.* Ericsson Technics No. 1, 1961, p. 12.

The conversation time for trunk calls is at present in Sweden about $T_0 = 0.02$ hr/day. In the sequel we assume $T_0 = 0.01$ hr/day.

With these assumptions the factor ξ can be calculated and the optimum reference equivalent can be read from *Fig. 3.2* (see *Table 3.2*).

Table 3.2. Optimum reference equivalent

Subscriber distribution	ξ	R_{max}	R_{99}
a short tail..........	0.15	51	33
b normal tail........	0.21	45	35
c long tail	0.27	42	37

It will be seen from the table that R_{max} is altogether too high and that the intelligibility on calls between the peripheries of the two networks is inadequate. The prolongation of conversation time at 50 db is 34 per cent (cf. *Fig. 2.1*). This means that special measures

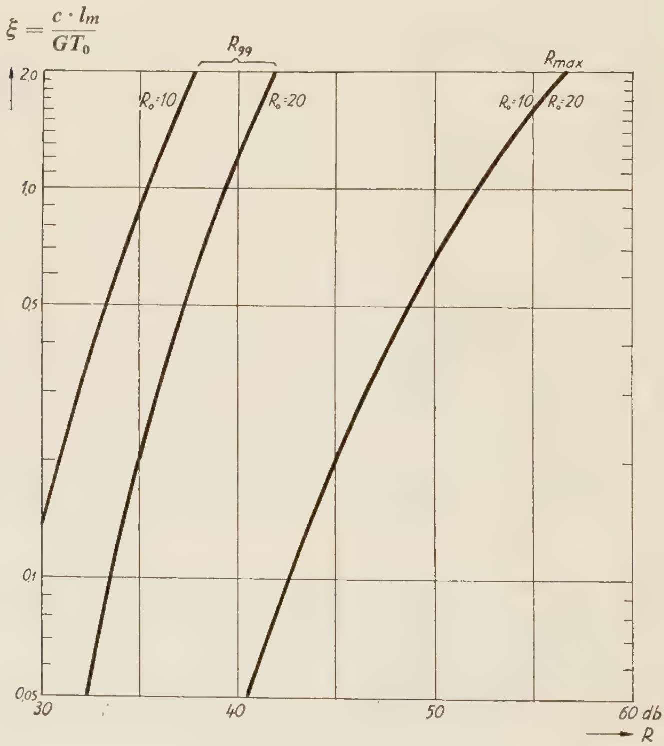

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Fig. 3.3. Relation between the factor ξ and R_{max} and R_{99} for $R_0 = 10$ and $R_0 = 20$. $L = 5$ km, normal case, $l_m = 1.54$ km.

should be taken for these subscribers, such as the installation of repeaters or of cables with larger dimensions of conductor. In the next chapter it is shown that such measures are warranted also on economic grounds. For the great majority of calls, however, the reference equivalent is considerably lower, as appears also from *Table 3.2*, in which R_{99} indicates the upper limit for the reference equivalent for 99 per cent of calls. The relation between ξ and R_{99} is also shown by *Fig. 3.2.*

Fig. 3.2 likewise shows the prolongation of conversation time, $\vartheta_m - 1$, for normal cases (*b*). It is interesting to note that this prolongation is of the same order of magnitude as the congestion in switches.

Fig. 3.3 shows R_{max} and R_{99} as functions of ξ for $R_0 = 10$ and $R_0 = 20$. It is apparent from the figure that R_{max} is practically as large for $R_0 = 10$ as for $R_0 = 20$, whereas R_{99} diminishes by about 2 db.

If the factor ξ, as before, is put $\xi = 0.21$, the values will be as in *Table 3.3.*

Table 3.3. Optimum reference equivalent

$L = 5$ km, $\xi = 0.21$, "normal" network, $l_m = 1.54$ km.

Reference equivalent	Reference equivalent independent of conductor diameter R_0	
	10 db	20 db
Maximum reference equivalent	45	45
Upper limit of reference equivalent for 99 per cent of calls	31	35
Mean reference equivalent	21	28

It is seen from this table that an improvement in characteristics of the telephone sets by, say, 1 db results approximately in

unchanged maximum reference equivalent

lowering by 0.4 db of the upper limit for the reference equivalent for 99 per cent of calls

lowering by 0.7 db of the mean reference equivalent

From this it is apparent that the greater part of any improvement in the transmission characteristics of telephone sets should be given to the subscribers.

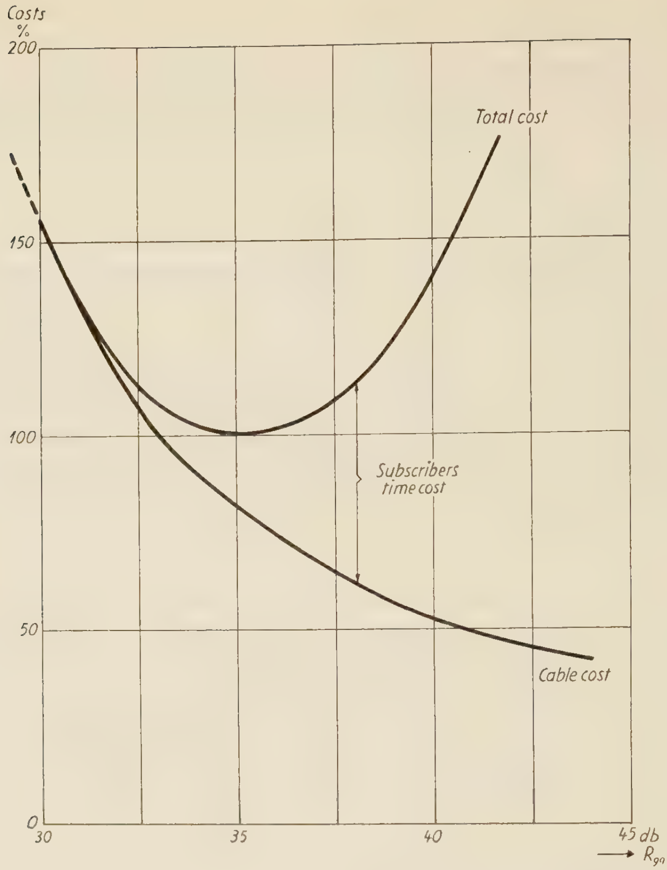

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Fig. 3.4. Character of cost minimum
$L = 5$ km $\qquad R_0 = 20$ db $\qquad \xi = 0.21$

In conclusion to this chapter, *Fig. 3.4* shows how the costs of the cable network and the prolongation of conversation time change with R_{99}.

The minimum is quite pronounced, and one sees that a change in the reference equivalent by $\pm$ 2 db from the optimal value increases the cost by about 5 per cent.

It will have been apparent from this chapter that the reference equivalent for subscribers on the periphery of a network, calculated on the basis of the overall prolongation of conversation time for all subscribers in the network, is very high, being of the order of 40—50 db in the numerical examples presented. So high a reference equivalent diminishes the intelligibility to a very great extent and results in a prolongation of conversation time by up to some 34 per cent. By way of comparison with this value, it may be mentioned that the

prolongation for 99 per cent of calls is not above 2 per cent and that the mean prolongation for all subscribers in the network is of the order of 0.003.

The reason why the peripheral subscribers have been so unmannerly treated is, of course, partly that subscribers at distances as great as 5 km from the exchange have presumably had cables of the same conductor diameter as in the centre of the area, partly that a very small proportion of the subscribers is generally situated on the periphery. No economic calculations are needed to decide whether special measures should be taken to provide better transmission conditions for these subscribers, since this is quite simply an absolute and imperative requirement.

At high values of reference equivalent the inconveniences to subscribers cannot be adequately described by means of measurements of the time a person takes to correctly understand a particular message.[1] For it must also be remembered that, in an actual telephone conversation, a prolongation of this nature has a cumulative effect and that conversation cannot, in fact, be conducted at all if the reference equivalent exceeds a given limit, which is roughly 55 db at normal room noise level.

One could take this into account by introducing a multiplier into the expression for prolongation of conversation time, ϑ, and instead of ϑ writing, for example,

$$\vartheta' = \vartheta \cdot \frac{\vartheta_\infty - 1}{\vartheta_\infty - \vartheta}$$

where ϑ_∞ corresponds to a reference equivalent at which intelligible conversation cannot take place. But even if such a correction could be made by expanding the measurements, this would still not be sufficient. For at high values of reference equivalent one must also take account of the risk of misunderstanding and of the economic value involved therein, of the subscribers' justified complaints, and of the costs and the loss of goodwill for the telephone administration. It is obviously difficult to try to translate inconveniences of all these kinds into monetary terms. But fortunately this is not necessary either, since our aim in this chapter is to establish an approximate upper limit for the reference equivalent.

The failure to attempt to evaluate the aforementioned circumstances has been one reason why this limit is too high.

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CHAPTER 4

The Reference Equivalent when Repeaters are Used on the Periphery of the Network

The large increase in conversation time for peripheral subscribers, which is a result of the calculations presented in the preceding chapter, poses the question whether it is profitable to take special measures on their behalf, such as the provision of cables with larger gauge conductors or of repeaters.

In the present chapter we shall confine our examination to the question of repeaters, using the same method as was criticized at the end of the last chapter. For this criticism loses its edge if the examination shows that a noticeably multiplicative effect of the prolongation of conversation time is improbable, and if the risk of misunderstanding and of complaints from subscribers need not be expected to be dependent on the change in transmission level within the domain relevant to the calculations. This will be found to be the case, at least with a large measure of plausibility.

In principle one should consider the following cost items:

cost of subscribers' cables

cost of repeaters

economic value of inconveniences to subscribers owing to prolongation of conversation time

and in the choice of conductor diameter, type of repeater, and area for which repeaters are to be provided, select those which result in a minimum aggregate cost.

For this purpose a knowledge is required of, among other things, how the cost of a repeater increases with the gain desired. The relation between cost and gain is not continuous, however, but the cost of a repeater increases discontinuously above given thresholds of gain.

For this reason it seems best to start with certain fixed values for the repeater cost. In such case it is clearly necessary to decide on fixed values for the diameter of the cable conductors as well.

We assume that two identical networks are interconnected and that the part of the reference equivalent which is independent of conductor diameter is R_0 db; also that repeaters are provided for subscribers at a distance of more than u km from the exchange. Without repeaters the reference equivalent for a circuit between such subscribers would exceed

$$R_u = (2\alpha + \alpha_m) \cdot u + R_0 \quad \text{db} \tag{4.1}$$

Assume also that the gain on circuits at a distance greater than u from the exchange can be brought up to a level such that the prolongation of conversation time will, on an average, be the same as for subscribers at a distance less than u and that the cost of this gain is q kr/subscriber.

In accordance herewith we determine the prolongation of conversation time for subscribers whose distances y and x km from the exchange fulfil the following inequalities:

$$\left.\begin{array}{c}(\alpha+\alpha_m)\cdot y+\alpha\cdot x+R_0>R_*\\ x\leq u\\ y\leq u\end{array}\right\} \tag{4.2}$$

The mean prolongation of conversation time, ϑ_m, for subscribers situated within this area, Ω', is calculated from the formula

$$\vartheta_m-1=\iint_{\Omega'} f(y)\cdot f(x)\cdot(\vartheta_{yx}-1)\cdot dy\cdot dx \tag{4.3}$$

The fraction, $\varkappa$, of the number of subscribers situated outside this area is

$$\varkappa=\int_u^L f(x)\cdot dx \tag{4.4}$$

The total costs for the subscribers due to the prolongation of conversation time in accordance with eq. 4.3 and to the provision of repeaters in accordance with eq. 4.4 is

$$2GT_0(\vartheta_m-1)+\varkappa\cdot q \tag{4.5}$$

in which, accordingly,

G = grade of service factor

T_0 = expected total duration of calls in hours initiated per subscriber per day under ideal transmission conditions

ϑ_m = mean prolongation factor

$\varkappa$ = the fraction of the number of subscribers with repeaters

q = the repeater cost, kr/subscriber

From eq. 4.5 it is possible to calculate u, and accordingly R_u, i.e. the maximum reference equivalent for a telephone circuit between the two networks, provided that the geographical distribution of the subscribers is known.

It is found that the maximum reference equivalent is dependent, firstly, on the relation between the value of the telephone calls and the repeater costs, while it is only slightly

affected by the character of the subscriber distribution. *Table 4.1* shows the upper limit for the reference equivalent under the assumptions indicated in the table.

Table 4.1. Upper limit for the reference equivalent between two subscribers in a national network after the introduction of repeaters

Maximum reference equivalent without repeaters 40—48 db
$G = 360{,}000$ kr $T_0 = 0.02, 0.01, 0.005$ hrs/day
Repeater cost $q = 20, 40, 80$ kr/subscriber $L = 5$ km
Subscriber distributions as in *Table 3.1* and *Fig. 3.1.*

$2GT_0$	Upper limit of reference equivalent db		
	$q = 20$	$q = 40$	$q = 80$
14,400	32	33	34
7,200	33	34	35
3,600	34	35	37

We note that the calculation of the values of maximum reference equivalent is based on

a high value for the maximum length of line, viz. 5 km

relatively low values of trunk traffic (in Sweden T_0 is $\simeq 0.02$ hr/day)

the highest values resulting from different assumptions concerning the geographical distribution

This means that the reference equivalent for a national circuit should be less than the tabulated values, perhaps about 33 db, if the repeater cost does not exceed 40 kr/subscriber. The calculations that have been made are, however, not sufficiently extensive to give a definite indication of the exact value of the reference equivalent. But they do afford support to the increasingly widespread opinion that the present recommendations should be changed and that the reference equivalent should be lowered.

CHAPTER 5

Principles for Division of a Given Overall Reference Equivalent between Sending and Receiving within a National Network

In this chapter we proceed from the assumption that the maximum permissible reference equivalent for a circuit between two subscribers has been fixed from the outset, and take as our task to divide this reference equivalent between sending and receiving so as to minimize the sum of the cable plant costs.

This formulation of the problem implies that we should no longer consider the value of the subscribers' time; for this may be said to have been done, at least to some extent, insofar as the overall reference equivalent has been established from the outset.

The reference equivalent from a subscriber to the terminal point of the international line is made up of the loss in local cables, junction cables, trunk cables, exchanges and telephone sets. This loss will hereafter be divided into two classes:

1. loss in local network
2. other losses

The loss in the local network is regarded as a function of the conductor diameter in the network, and other losses are regarded as parameters which assume different values according to the parts of the network the connection has to pass through. Under these assumptions the problem can be given the following simple formulation. Determine

R_{sm} = permissible maximum sending reference equivalent

R_{rm} = permissible maximum receiving reference equivalent

so that the costs for a given network configuration, with given values of loss in junction cables, telephones and exchanges, is a minimum provided that

$$R_m = R_{sm} + R_{rm} = \text{constant}$$

For this purpose we introduce the following notations:

R_s = sending reference equivalent, db

R_{sm} = maximum permissible value, sending, db

R_r = receiving reference equivalent, db

R_{rm} = maximum permissible value, receiving, db

$\alpha = \frac{c_1}{z}$ = a.c. loss, db/km ($c_1 = 0.58$)

$\alpha_m = \frac{c_1}{z^2}$ = transmitter feed loss, db/km ($c_2 = 0.19$)

z = conductor diameter in mm

R_{0s} – loss in telephone set, junction cables, exchanges, incl. tolerances for sending

R_{0r} = ditto for receiving

R_{0i} = loss on national trunk line

$R_0 = R_{0s} + R_{0r} + R_{0i}$

If L denotes the longest subscriber's line in a given network, the conductor gauge in the network is determined by the fulfilment of the following inequalities:

$$R_s = (\alpha + \alpha_m) \cdot L + R_{0s} \leq R_{sm} \tag{5.1}$$

$$R_r = \alpha \cdot L + R_{0r} \leq R_{rm} \tag{5.2}$$

If the values of R_{sm} and R_{rm} are fixed, then, as is apparent from these expressions, the sending reference equivalent will be determinative of the conductor gauges as soon as

$$R_{sm} - R_{rm} < \alpha_m \cdot L + \Delta R_{sr} \tag{5.3}$$

while the receiving reference equivalent will be determinative in the contrary case, viz.

$$R_{sm} - R_{rm} > \alpha_m \cdot L + \Delta R_{sr} \tag{5.4}$$

with

$$\Delta R_{sr} = R_{0s} - R_{0r} \tag{5.5}$$

ΔR_{sr} is the difference between the sending and receiving reference equivalents of the telephone set.

Eq. 5.1 and 5.2 give the same value, z_0, of the conductor diameter when

$$z_0 = \sqrt{\frac{c_2 L}{R_{sm} - R_{rm} - \Delta R_{sr}} = \frac{c_1 L}{R_{rm} - R_{0r}}} \tag{5.6}$$

Hence the values of R_{rm} and R_{sm} can be calculated which give the smallest conductor diameter in a given network, and so the smallest cost for the cable plant:

$$R_{rm} = R_{0r} + \frac{c_1{}^2}{c_2} \cdot L \left[\sqrt{1 + \frac{c_2}{c_1}(R_m - \Delta R_{sr} - 2 R_{0r})} - 1 \right] \tag{5.7}$$

$$R_{sm} = R_m - R_{rm} \tag{5.8}$$

As soon as one diminishes R_{rm} in relation to R_{sm}, then obviously R_{rm} will determine the gauge of conductor, in the contrary case R_{sm} will be the determining factor. Under either alternative the conductor diameter in the networks will increase. This is illustrated in *Fig. 5.1*, which shows z^2 as function of R_{rm} for different values of R_0 at $R_{sr} = 5$ and $L = 5$ km.

As is seen, the cable plant costs which are directly proportional to z^2 increase fairly quickly when the curve shifts to the right or left of the minimum point.

If we assume, as an example, that with certain characteristics of the telephone sets, R_0 in a given network cannot be below 15.8 db nor above 31.8 db, we immediately obtain the following limits for the receiving and sending reference equivalents on the assumption that R_m is 31.2 db:

$$9.3 < R_{rm} < 12.4$$

$$21.9 < R_{sm} < 18.8$$

These limits are altogether too wide, and an attempt will now be made to narrow them down.

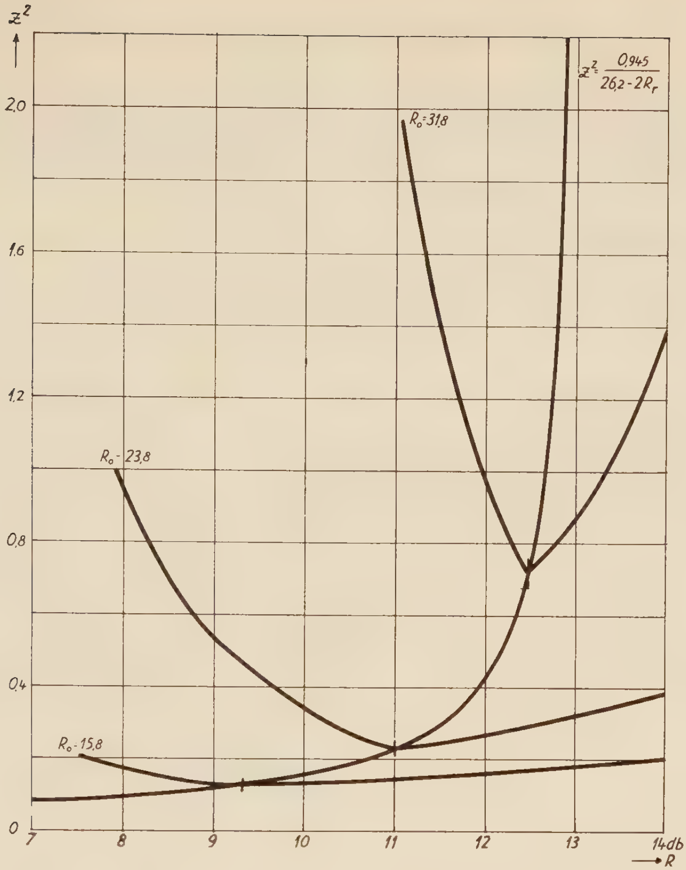


Fig. 5.1.

$z^2 = f(R_{rm})$

$R_{sm} + R_{rm} = 31.2$ db, $R_{0s} - R_{0r} = 5$ db.

$L = 5$ km

The part of the cable plant costs which changes with the conductor diameter can, as is known, be written

$$k_n = c \cdot l_m \cdot z^2 \tag{5.9}$$

where

c = a constant, kr/km mm²

l_m = the mean length of line in the network, km

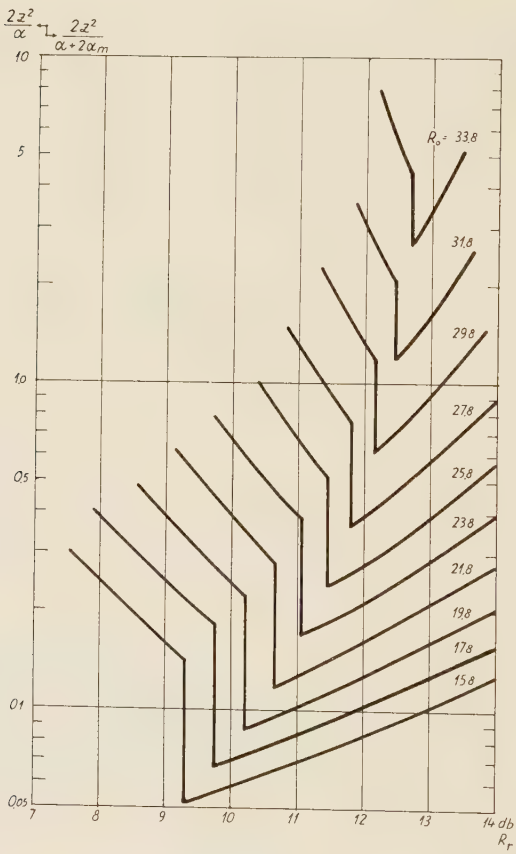
This gives

$$\frac{dk_n}{dR_r} = 2c \cdot l_m \cdot \frac{dz}{dR_r} \tag{5.10}$$

where $\frac{dz}{dR_r}$ is determined from eq. 5.1 as soon as the sending reference equivalent is the determining factor for the conductor gauge, and from eq. 5.2 as soon as the receiving reference equivalent is the determining factor. In the former case we speak of cost derivation to the right of the minimum point, in the latter case of cost derivation to the left. From eq. 5.1, 5.2 and 5.10 we immediately obtain

Derivation to the right

$$\left|\frac{dk_n}{dR_s}\right| = c \cdot \frac{l_m}{L} \cdot \frac{2z^2}{\alpha + 2\alpha_m} \tag{5.11}$$



X 11267

Fig. 5.2.

$\left|\frac{dk_n}{dR_r}\right|$ and $\left|\frac{dk_n}{dR_s}\right|$ as function of R_r

$R_r + R_s = 31.2$ db

Derivation to the left

$$\left|\frac{dk_n}{dR_r}\right| = c \cdot \frac{l_m}{L} \cdot \frac{2z^2}{\alpha} \tag{5.12}$$

Fig. 5.2 shows the course of $\frac{2z^2}{\alpha + 2\alpha_m}$ and $\frac{2z^2}{\alpha}$ as function of R_r on the assumption that $R_r + R_s = 31.2$ and under different assumptions concerning R_0.

By means of this figure one can calculate how a given overall reference equivalent, $R_s + R_r = 31.2$, shall be divided between sending and receiving on the assumption that one knows how the number of subscribers and the costs of the subscribers' lines are distributed in relation to trunk circuits with different values of R_0.

The result of such a calculation will differ according to the character of the national network. In networks serving a dense population the number of areas having a small value of R_0 will be large in comparison with those having a large value of R_0. The reverse will apply in a sparsely populated country. By way of example we assume that a national network can be represented by the following values of R_0 and with the following weights:

R_0	15.8	19.8	23.8	27.8	31.8
Weight	16	8	4	2	1

In this case a calculation with the aid of *Fig. 5.2* yields the result that the overall reference equivalent, 31.2 db, shall be divided as follows:

maximum sending reference equivalent	19.4 db
maximum receiving reference equivalent	11.8 db
Total	31.2 db

If, instead, we assume the weights to be

81 : 27 : 9 : 3 : 1

we get

maximum sending reference equivalent	20.0 db
maximum receiving reference equivalent	11.2 db
Total	31.2 db

Finally, if it is assumed that all networks have the same weights, we get

maximum sending reference equivalent	18.8 db
maximum receiving reference equivalent	12.4 db
Total	31.2 db

Unfortunately the data are at present lacking for an exact calculation of national networks. It seems probable, however, that the actual conditions would generally be such that

$$20 > R_s > 18.8$$

$$11.2 < R_m < 12.4$$

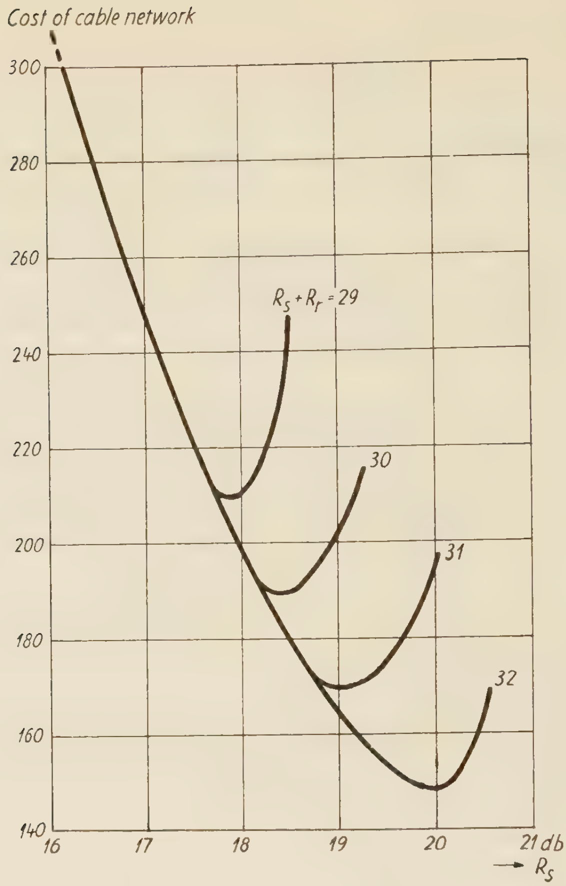


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Fig. 5.3. Costs of a cable network in a special case as function of the sending reference equivalent at different values of $R_s + R_r$.

Examination of the character of the minimum shows that the cost rises considerably, even for comparatively insignificant deviations from the optimum distribution (*cf. Fig. 5.3*).

It follows that *the division of the overall reference equivalent should not be fixed once and for all, but be decided in relation to the character of each national network.**

* According to the latest recommendations of C.C.I.T.T. (New Delhi 1960) such a division of the overall reference equivalent can be made.

ANNEX 1

The Reference Equivalent under the Assumption that the Prolongation Factor for the Peripheral Subscribers is Used as the Basis for Calculating the Time Losses of All Subscribers

The assessment of the inconveniences to subscribers due to unsatisfactory transmission has in this paper been consistently based on the mean prolongation factor for conversations between networks. If we, instead, set up the hypothesis that the assessment should be based on the inconveniences to the subscribers on the periphery of the networks, the advantage is gained at least that the problem is greatly simplified and can be dealt with analytically. Such a hypothesis, at all events, offers a ready means for a preliminary survey of the problem. And owing to the often very pronounced tail of the geographical distribution of the subscribers, it gives a preliminary idea of the maximum reference equivalent for the great majority of calls, though without, of course, being able to specify exactly how great a proportion this majority represents.

If we adopt this procedure, we need not worry about the geographical distribution. And, on the assumption that two identical networks are linked by a trunk circuit, the cost expression to be minimized is, quite simply,

$$c \cdot l_m \cdot z^2 + 2GT_0(e^{\gamma(R-R_*)^2} - 1) \qquad (1)$$

in which, as before,

c = a constant, kr/km mm^2

l_m = the mean length of line in the networks, km

z = the conductor diameter, mm

G = the grade of service factor

T_0 = expected total duration of calls in hours initiated per subscriber per day under ideal transmission conditions

ϑ = $e^{\gamma(R-R_*)^2}$ the prolongation factor for calls at a reference equivalent R

R = $(2\alpha + \alpha_m) L + R_0$

$\alpha = \dfrac{c_1}{z} \qquad \alpha_m = \dfrac{c_2}{z^2}$

c_1, c_2 = constants

L = longest subscriber line in the network, km

R_0 = the part of the reference equivalent which is independent of conductor diameter

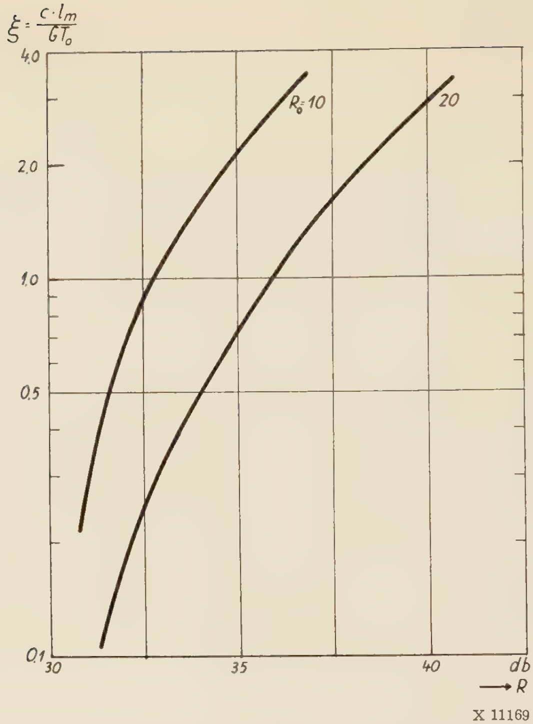

Fig. 1.

$$\xi = 4\gamma L \cdot (R - R_*) \cdot \frac{\alpha + \alpha_m}{z^2} \cdot e^{\gamma (R - R_*)^2}$$

$R_* = 30$ db, $L = 5$ km, $\gamma = 7.34 \cdot 10^{-4}$, $R_0 = 10, 20$ db, $R = (2\alpha + \alpha_m) \cdot L + R_0$

A necessary condition for expression 1 being a minimum is clearly

$$\xi = \frac{c \cdot l_m}{GT_0} = 4\gamma \cdot L \cdot (R - R_*) \cdot \frac{\alpha + \alpha_m}{z^2} \cdot e^{\gamma (R - R_*)^2} \qquad (2)$$

with

$$R = (2\alpha + \alpha_m) \cdot L + R_0$$

Fig. 1 shows the relation between ξ and R for $L = 5$ km and $R_0 = 10$ and 20 db.

In this case, of course, the reference equivalent will be considerably lower than on the basis of the mean prolongation factor ϑ_m according to eq. 3.2. Comparison with the aid of the subscriber distribution in *Fig. 3.1*, "normal case", shows that this very accurately portrays the reference equivalent for about *96 per cent* of calls at $0.1 < \xi < 1.0$, calculated on the basis of the mean prolongation of conversation time.

ANNEX 2

The Reference Equivalent on the Assumption that the Subscriber Distribution is Rectangular

As a second extreme case it may be of interest to calculate the reference equivalent on the assumption that the subscriber distribution is rectangular, which means that it can be written $f(x) = \frac{1}{L}$, $(0 < x < L)$. For two identical networks linked by a trunk circuit, the cost expression to be minimized in this case is clearly

$$c \cdot l_m \cdot z^2 + 2GT_0 \cdot (\vartheta_m - 1) \qquad (1)$$

Assuming that we write $\vartheta \approx 1 + \gamma(R - R_*)^2$, which can be done with good approximation so long as $R < 40$ db, this expression can be written

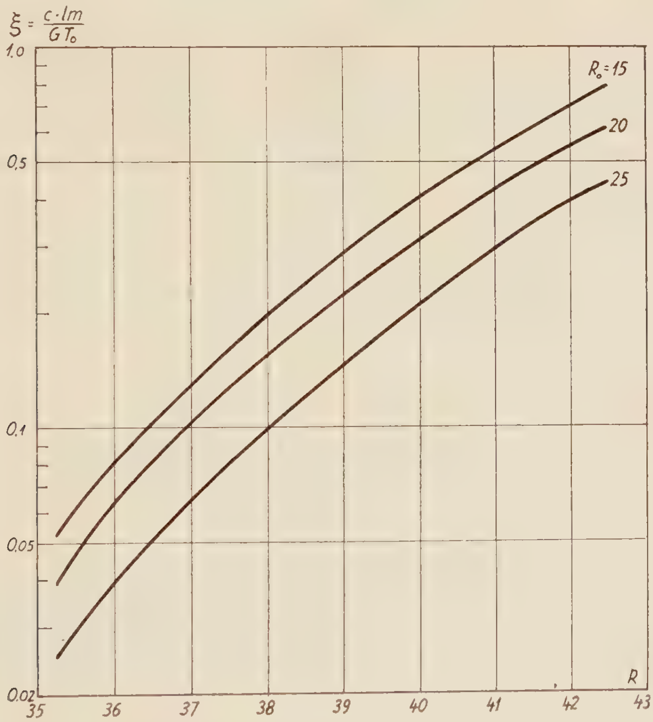

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Fig. 1. The reference equivalent for $R_0 = 15$, 20, 25 db with rectangular subscriber distribution as function of

$$\xi = \frac{c \cdot l_m}{GT_0}, \quad L = 5 \text{ km}.$$

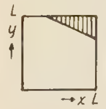
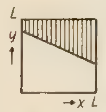
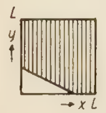
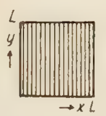
$$c \cdot l_m \cdot z^2 + 2GT_0 \cdot \frac{\gamma}{L^2} \iint_{\Omega} (R - R_*)^2 \cdot dy \cdot dx \qquad (2)$$

with

$$R = (\alpha + \alpha_m) \cdot y + \alpha \cdot x + R_0 \qquad (3)$$

Under this assumption both $\vartheta_m - 1$ and $\xi = -\frac{1}{z} \cdot \frac{\partial \vartheta_m}{\partial z}$ can be calculated explicitly. The result is shown in *Table 1* for the four cases of integration which can occur.

Table 1.

Integration domain Ω	$\dfrac{\vartheta_m - 1}{\dfrac{\gamma}{12 B_s B_r}}$	$-\dfrac{\dfrac{1}{z} \cdot \dfrac{\partial \vartheta_m}{\partial z}}{\dfrac{\gamma}{12 (B_s B_r)^2} \cdot \dfrac{B_r}{z^2}}$
 1 $B_s < \Delta R$	$(R_m - R_*)^4$	$[8 B_s^2 - (R_m - R_*)(3 B_s - B_r)](R_m - R_*)^3$
 2 $B_s < \Delta R < B_s$	$(R_m - R_*)^4$ $- (B_s - \Delta R)^4$	$[8 B_s^2 - (R_m - R_*)(3 B_s - B_r)](R_m - R_*)^3$ $- [4 B_s (2 B_s - B_r) - (B_s - \Delta R)(3 B_s - B_r)](B_s - \Delta R)^3$
 3 $0 < \Delta R < B_r$	$(R_m - R_*)^4$ $- (B_s - \Delta R)^4$ $- (B_r - \Delta R)^4$	$[8 B_s^2 - (R_m - R_*)(3 B_s - B_r)](R_m - R_*)^3$ $- [4 B_s (2 B_s - B_r) - (B_s - \Delta R)(3 B_s - B_r)](B_s - \Delta R)^3$ $- [4 B_s B_r - (B_r - \Delta R)(3 B_s - B_r)](B_r - \Delta R)^3$
 4 $\Delta R < 0$	$(R_m - R_*)^4$ $- (B_s - \Delta R)^4$ $- (B_r - \Delta R)^4$ $+ (\Delta R)^4$	$[8 B_s^2 - (R_m - R_*)(3 B_s - B_r)](R_m - R_*)^3$ $- [4 B_s (2 B_s - B_r) - (B_s - \Delta R)(3 B_s - B_r)](B_s - \Delta R)^3$ $- [4 B_s B_r - (B_r - \Delta R)(3 B_s - B_r)](B_r - \Delta R)^3$

$\vartheta_m - 1 = \iint_{\Omega} f(y) \cdot f(x) \cdot (\vartheta_{yx} - 1) \cdot dy \cdot dx$ and $\xi = -\frac{1}{z} \cdot \frac{\partial \vartheta_m}{\partial z}$ under condition that $f(x) = f(y) = \frac{1}{L}$ and $\vartheta - 1 \simeq \gamma (R - R_*)^2$

$R_m = (2\alpha + \alpha_m) L + R_0$
$B_s = (\alpha + \alpha_m) L$
$B_r = \alpha L$
$\Delta R = R_* - R_0$

$\gamma = 7.34 \cdot 10^{-4}$
$R_* = 30$ db
$\xi = \dfrac{c \cdot l_m}{GT_0}$

Fig. 1 shows the result of a calculation for $R_0 = 15$, 20 and 25 db.

With a rectangular distribution the mean length of line in the network will be $\frac{L}{2} = 2.5$ km which, with the same data as used for the construction of *Table 3.2* in chapter 3, gives $\xi = 0.34$. The corresponding maximum reference equivalent is $R_{max} = 40$ db for $R_0 = 20$ db. R_{99} is therefore equal to 38 db. Networks with rectangular subscriber distribution and maximum length of line as great as $L = 5$ km, mean length of line $l_m = 2.5$ km, are probably very rare—at all events no such "heavy" network was encountered among the material used for G. Lind's study. The above results, therefore, lend additional support to the view that a reference equivalent of 40 db is too high from the economic aspect.

ANNEX 3

Simplified Approximative Method for Determination of Mean Prolongation Factor, ϑ_m

The work of numerical calculation can be greatly simplified if one approximately puts

$$R(x, y) = \frac{R_{yx} + R_{xy}}{2} = \left(\alpha + \frac{\alpha_m}{2}\right) \cdot (y + x) + R_0 \tag{1}$$

This enables one to convert the double integral

$$\vartheta_m - 1 = \iint_{\Omega} f(y) \cdot f(x) \cdot (\vartheta_{yx} - 1) \cdot dy \cdot dx \tag{2}$$

into a simple integral

$$\vartheta_m - 1 = \int_{\frac{R_* - R_0}{\alpha + \frac{-\alpha_m}{2}}}^{2L} \varphi(u) \cdot (\vartheta_u - 1) \cdot du \tag{3}$$

by putting $u = x + y$ as new variable and convoluting the frequency distributions $f(x)$ and $f(y)$ to a new frequency function $\varphi(u)$ which expresses the probability that the sum of the distances to the respective exchanges will be u. In this way the work of numerical calculation is reduced to about one-tenth if the material is divided into ten classes.

The frequency function for "normal cases" at $L = 5$ km, drawn from the data given in *Table 3.1*, column *b*, is shown in *Fig. 1*, which also shows the twofold convolution of this frequency function. And lastly a normal distribution is shown (curve 3) with the same mean value and standard deviation as in curve 2.

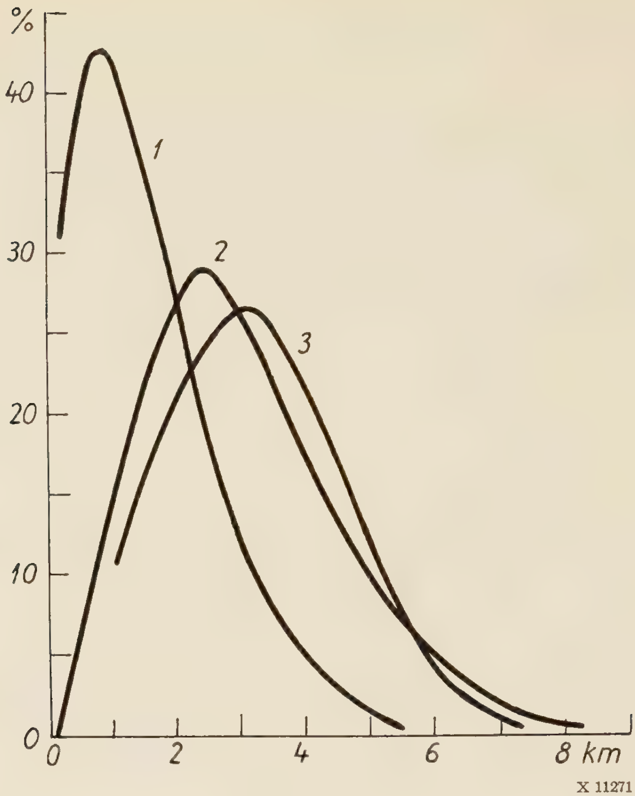


Fig. 1.

1. Frequency function for normal case
2. Twofold convolution of this frequency function
3. Normal distribution with the same mean value and standard deviation as in curve 2

The agreement between curves 2 and 3 is in this case not very good. But if we represent the frequency function (curve 2) by a normal distribution, the expression for ϑ_m can be reformed so as to be readily calculable from normal distribution tables. For if the mean value of the frequency function (no. 2, *Fig. 1*) is μ and the standard deviation σ, one can reform the expression (cf. eq. 3)

$$\vartheta_m - 1 = \int\limits_{u=l=\frac{R_* - R_0}{\alpha + \frac{\alpha_m}{2}}}^{2L} \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2\right) \cdot \left\{\exp\left[\gamma\left(\left(\alpha + \frac{\alpha_m}{2}\right) \cdot u - (R_* - R_0)\right)^2\right] - 1\right\} \cdot du \tag{4}$$

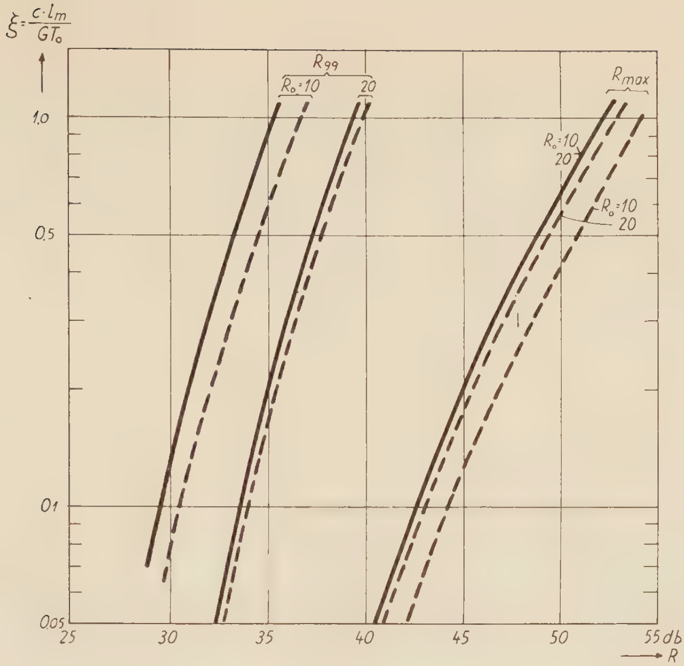


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Fig. 2. Relation between ξ and R.

——— Exact calculation, cf. fig. 3.2.

- - - - - - Approximate calculation according to eq. 3 in this Annex.

into

$$\vartheta_m - 1 = \sqrt{\frac{1}{1 - 2\beta\sigma^2}} \cdot \exp\left(\frac{\beta(\mu - l)^2}{1 - 2\beta\sigma^2}\right) \cdot \int_{t_1}^{t_2} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}t^2\right) \cdot dt -$$

$$- \int_{l}^{2L} \frac{1}{\sigma \cdot \sqrt{2\pi}} \exp\left[-\frac{1}{2} \cdot \left(\frac{x - \mu}{\sigma}\right)^2\right] \cdot du \qquad (5)$$

with

$$t = \frac{u - \mu \cdot \dfrac{1 - 2\beta\sigma^2 \cdot \dfrac{l}{\mu}}{1 - 2\beta\sigma^2}}{\sigma\sqrt{\dfrac{1}{1 - 2\beta\sigma^2}}}$$

$$\beta = \gamma \cdot \left(\alpha + \frac{\alpha_m}{2}\right)^2 \qquad l = \frac{R_* - R_0}{\alpha + \dfrac{\alpha_m}{2}}$$

3—106690

Under the obvious condition that

$$\sigma < \sqrt{\frac{1}{2\beta}} \tag{6}$$

this expression can be used for calculation of ϑ_m for, for example, a given value of μ and different values of σ, and in this way one can acquire a good survey of the limits within which ϑ_m should lie for different types of geographical distribution.

Having found in this way an approximately explicit expression for ϑ_m, it is naturally possible also to calculate explicitly the expression

$$\xi = \frac{c \cdot l_m}{GT_0} = -\frac{1}{z} \cdot \frac{\partial \vartheta_m}{\partial z} \tag{7}$$

This will be rather complicated, however, since the conductor diameter is included both in all factors and in the limits of the integral. As before, therefore, eq. 7 can best be calculated by interpolation with the aid of the results obtained with eq. 5.

By way of example, *Fig. 2* shows the function $\xi = -\frac{1}{z} \cdot \frac{\partial \vartheta_m}{\partial z}$ calculated by this simplified approximate method. This figure is based on the same data as is the calculation of *Fig. 3.2*.

Acknowledgements

The author wishes to express his thanks to Dr. C. Jacobæus, Telefonaktiebolaget L M Ericsson, for his encouragement of the investigations in this field, to Mr. H. Hansson who conducted the tests regarding the prolongation of conversation time, to Mr. G. Lind who made the statistical analysis of the results and furthermore developed a model for the geographical distribution of subscribers, to Mr. J. Pyddoke and Mr. G. Lind who read the proofs and gave valuable advice, and finally to Mrs. T. Miemois who carried out the comprehensive numerical calculations.

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On the Dependence between the Two Stages in a Link System

BY

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LME 807

A two-stage link system is considered, with one link in every cross-point in the second stage and with every route consisting of one outlet column. The effects of the dependence between the link stage and the outlets are analytically described as deformations of the original traffic distributions which would prevail if there were no interaction between the two parts of the link system. The deformations are caused by the blocked calls and are only of importance when the congestion is high, as happens, for example, when a link system is used for alternative routing.

Following the same line of thought as in two earlier papers (1956 and 1957), but modified so as to take into account larger link systems of a size occurring in practice, expressions are derived for the distributions for the link columns, the outlet columns and the number of occupied pairs, as well as for time and call congestion.

For evaluation of the formulae, which of course are approximate, an iteration process was applied and carried out in the Swedish Computer BESK.

The evaluations have been compared with artificial traffic trials on link systems. It is found that the analytical explanation presented covers the major part of the dependence effect. The analytical explanation may facilitate a closer approach to a practical solution of the problem of dimensioning link systems under conditions of high congestion.

Paper read at The Third International Teletraffic Congress, Paris, September 11—16, 1961

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Contents

Page

INTRODUCTION 185

CHAPTER 1 *General Conditions for a Two-Stage Link System* 186

CHAPTER 2 *Method* 189

2.1 Possible Methods 189

2.2 Derivation of the Method 191

2.3 Evaluation of the Method 202

2.4 Simplified Case 206

2.5 Modified Method 214

CHAPTER 3 *Comparison with Results of Artificial Traffic Measurements* 221

3.1 Comparison No. 1 225

3.2 Comparison No. 2 228

3.3 Comparison No. 3 230

3.4 Comparison No. 4 238

3.5 Result of Comparisons as Regards the Calculated Distributions . . . 247

3.6 Result of Comparisons as Regards the Calculated Congestion Values . 254

CONCLUSIONS 256

ACKNOWLEDGEMENTS 258

BIBLIOGRAPHY 258

Introduction

For calculation of the congestion in link systems, methods of sufficient accuracy for most cases of practical importance have been given by Jacobaeus (1947 and 1950). His methods were further improved by Jensen (1952_{I} and 1952_{II}), Rodenburg (1953), Le Gall (1956, 1957 and 1959), and others. The methods for the practical dimensioning of link systems are based on the assumption that the stages in a link system can be considered as being independent of one another. From measurements on link systems carrying real traffic (see, e. g., Jacobaeus (1950), and Elldin, Jacobaeus, von Sydow (1954)), as well as on link systems with artificial traffic simulated in an electronic computer (Wallström 1958 and 1961), it has been found that the methods outlined by Jacobaeus in 1950 are sufficiently accurate when the congestion is low. It has also been found that the assumption of independence between the stages generally gives a certain safety margin compared with the ideal conditions existing for artificial traffic. This safety margin may perhaps explain the very good accuracy during the less ideal conditions which can be expected to prevail for real telephone traffic.

There are, however, some cases of increasing practical importance in which the safety margin following from the independence assumption is unnecessarily large, namely when a link system is used for alternative routing. In this case the congestion values on the direct routes are of such a magnitude (5—50%) that too large a safety margin may lead to uneconomical overprovision of lines in the network. Consequently, when the congestion is large, there is reason to introduce some modifications into the ordinary practical methods of calculation. This can be done by taking into account the dependence phenomena. Before such modifications can be made, a logical explanation of how the dependence influences the traffic conditions in a link system is necessary. In order to facilitate further steps toward a practical solution of the problem, this paper aims at giving an analytical explanation of the dependence in a simple two-stage link system.

In two earlier papers (1956 and 1957), the author considered the possibility of taking into account the dependence in a two-stage link system by employing the method of equations of state. As can be seen from comparisons with artificial traffic trials (Wallström 1958), the method shows very good consistency with the measured results. The small differences that still exist between calculated and measured values can be explained by the fact that the equations of state only deal with the conditions for one link column and one outlet column, and can consequently only approximately take into account the influence from the rest of the link system. In spite of these approximations, the equations of state method seems to explain the greater part of the dependence phenomena. However, the method cannot be applied without difficulty on link systems of a size that is normal in practice, since the number of possible states increases too fast with the size. It will be very difficult, therefore, or practically impossible, to check whether conclusions drawn for a small link system have the same significance for a large one. Hence it is desirable to have an evaluation method that can be applied also to large link systems. Such a method is outlined in the sequel. Since the method of equations of state seems sufficiently accurate, the same line of

thought is used for description of the dependence, but the method of calculation is changed so as to be applicable to a two-stage link system of ordinary size. Since results are available from measurements with artificial traffic, simulated in the Swedish computer BESK, (WALLSTRÖM 1961), an opinion can immediately be formed as to the accuracy of the method. This is done in the last chapter of this study.

CHAPTER 1

General Conditions for a Two-Stage Link System

A two-stage link system as shown in *Fig. 1.1* is considered. It consists of $n \cdot k$ inlets in the A-stage, $m \cdot k$ links in the B-stage and $m \cdot v$ outlets in C. The link system, with k AB-columns and v outlet columns in C, is assumed to act as a closed unit, *i.e.* only calls initiated from the $m \cdot k$ inlets will occupy the links in the B-stage and the outlets in C.

All calls from inlet to outlet will be set up by conditional selection, *i.e.* an inlet in A, a link in the corresponding column in B and an outlet in C will be occupied simultaneously.

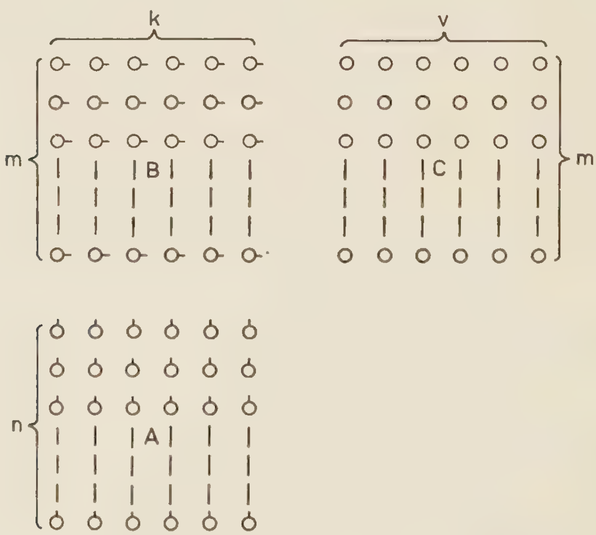


Fig. 1.1. A two-stage link system with inlets in stage A, links in stage B and outlets in C.

Calls that cannot be set up to the desired outlet or outlets will be immediately rejected and will occupy neither inlets nor links. Consequently, the link system is assumed to act under the assumption of Lost Calls Cleared, and the A-stage, the B-stage and the outlets in C will always have the same number of simultaneous occupations. Consequently, every successful call will occupy one inlet, one link in the B-column corresponding to the inlet, and one outlet in the row corresponding to the link.

The link system can at most handle a number of simultaneous calls equal to the smaller of $n \cdot k$ and $m \cdot \mathrm{v}$.

When there is no risk of confusion, the B-columns in the link system of *Fig. 1.1* will be referred to as $\mathbf{B}_1, \mathbf{B}_2 \ldots, \mathbf{B}_k$, or generally $\mathbf{B}_i$ $(i = 1, 2, \ldots k)$.

Analogously, the outlet columns will be referred to as $\mathbf{C}_j$, $(j = 1, 2, \ldots, \mathrm{v})$.

Number of occupations in the link system

For the number of simultaneous occupations at a point of time, t, the following notations are used:

q_i = total number of simultaneous occupations in $\mathbf{B}_i$, $(1 \leq i \leq k)$
r_j = total number of simultaneous occupations in $\mathbf{C}_j$, $(1 \leq j \leq \mathrm{v})$
s_{ij} = total number of simultaneous occupations in $\mathbf{B}_i$ that are connected to $\mathbf{C}_j (1 \leq i \leq k, 1 \leq j \leq \mathrm{v})$

For q_i, r_j and s_{ij} the following limits are valid for all possible values of i and j:

$$\left.\begin{array}{ll} 0 \leq s_{ij} \leq q_i \leq n & \text{if } n \leq m \\ 0 \leq s_{ij} \leq q_i \leq m & \text{if } n \geq m \\ 0 \leq s_{ij} \leq r_j \leq m & \end{array}\right\} \quad (1.1)$$

By definition the following relations between q_i, r_j and s_{ij} are true at any point of time:

$$\left.\begin{array}{l} q_i = \sum_{j=1}^{\mathrm{v}} s_{ij} \\ \\ r_j = \sum_{i=1}^{k} s_{ij} \end{array}\right\} \quad (1.2)$$

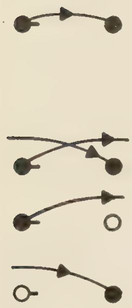
Since the A-stage, the B-stage and the outlets in C all have the same number of simultaneous occupations, the following relation between q_i, r_j and s_{ij} is fulfilled at any point of time:

$$\sum_{i=1}^{k} q_i = \sum_{j=1}^{\mathrm{v}} r_j = \sum_{i=1}^{k} \sum_{j=1}^{\mathrm{v}} s_{ij} \quad (1.3)$$

Number of occupied pairs

To define the possibility of connecting from an AB-column to an outlet column in C, it is necessary to introduce a fourth type of parameter, the number of occupied pairs. We therefore introduce p_{ij} = the number of pairs, consisting of a link in $\mathbf{B}_i$ and an outlet in $\mathbf{C}_j$, which at a point of time, t, cannot be used for connecting from $\mathbf{B}_i$ to $\mathbf{C}_j$, $(1 \leq i \leq k, 1 \leq j \leq v)$.

An occupied pair, (ij), can be constituted in four different ways:



1. The link in $\mathbf{B}_i$ is connected to an outlet in $\mathbf{C}_j$ (internal connection).
2. The link in $\mathbf{B}_i$ is connected to an outlet in another C-column and
   - 2.1 the corresponding outlet in $\mathbf{C}_j$ is occupied by a link in another B-column
   - 2.2 the corresponding outlet in $\mathbf{C}_j$ is free.
3. The link in $\mathbf{B}_i$ is free, but the outlet in $\mathbf{C}_j$ is occupied by a link in another B-column.

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The relation between the parameters p_{ij}, q_i, r_j and s_{ij} will be apparent from *Fig. 1.2*, where blacked-out symbols denote occupied devices.

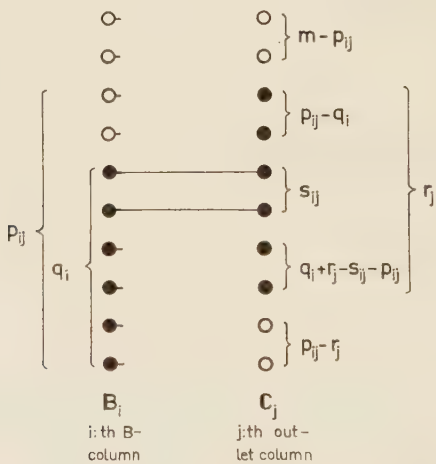

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Fig. 1.2. Example of occupation of the i:th B-column ($\mathbf{B}_i$) and the j:th outlet column ($\mathbf{C}_j$). Blacked-out symbols denote occupied devices.

The following conditions are valid at every point of time t:

$$\left.\begin{array}{l} 0 \le p_{ij} \le m \\ 0 \le s_{ij} \le q_i \le p_{ij} \\ 0 \le s_{ij} \le r_j \le p_{ij} \\ p_{ij} \le q_i + r_j - s_{ij} \end{array}\right\} \quad (1.4)$$

It is to be observed that the assumptions hitherto do not define the order of hunting for a free pair. The relation between the number of inlets per A-column, n, and the number of links per B-column, m, may be arbitrary, i.e. $n \le m$, or $n > m$.

Furthermore, no assumption is made as to how the outlets in C are divided between different routes, which may consist of one or more C-columns. In this sense, the parameters p_{ij}, q_i, r_j and s_{ij} are more general than is necessary for the following treatment of the link system.

For the calculations made in this study the following additional assumptions are used:

1. Random hunting for a free pair, i.e. all $m - p_{ij}$ free pairs (i, j) have the same probability of being chosen for a call from $\mathbf{B}_i$ to $\mathbf{C}_j$.
2. Every outlet column $\mathbf{C}_j$ represents one route.

These two additional assumptions, and especially the latter, are not necessary if a more general treatment of the link system is intended than is the purpose of this study. It seems quite possible to use the same parameters for characterizing a link system in which, for example, some routes are used only as second choice routes.

CHAPTER 2

Method

2.1 Possible Methods

The general state in the two-stage link system shown in *Fig. 1.1* is fully defined by the parameters p_{ij} and s_{ij} ($i = 1, 2, \ldots, k, j = 1, 2, \ldots, v$). In fact, the general state can be expressed by a little less than $2kv$ parameters, since a few of them are fully defined by the others.

The probability of the general state for the whole link system can be written as

$$[p, s] = [p_{11}, p_{12}, \ldots p_{kv}, s_{11}, s_{12} \ldots s_{kv}] \tag{2.1.1}$$

and be calculated by ordinary equations of state. Today, however, the possibility of making

evaluations is limited to small link systems, since the number of unknowns increases very fast with the size of the link system. This is apparent from the following reasoning. Since

$$0 \leq s_{ij} \leq p_{ij} \leq m$$

every pair of parameters (p_{ij}, s_{ij}) can assume

$$\frac{(m+1) \cdot (m+2)}{2}$$

different values if $n \geq m$, and

$$\frac{(n+1) \cdot (n+2)}{2} + (m-n) \cdot (n+1)$$

different values if $n \leq m$.

Since there are altogether $k \cdot \mathrm{v}$ pairs (p_{ij}, s_{ij}), the number of unknowns will at most be

$$k \cdot \mathrm{v} \cdot \frac{(m+1) \cdot (m+2)}{2} \qquad (n \geq m)$$

$$k \cdot \mathrm{v} \cdot \left\{\frac{(n+1) \cdot (n+2)}{2} + (m-n) \cdot (n+1)\right\} \qquad (n \leq m)$$

There will be a certain reduction of the above number of possible states since the s_{ij} and p_{ij} must satisfy the conditions given in (1.1), (1.3) and (1.4). But there are still too many unknowns to be handled in an electronic computer. For instance, for $n \geq m$, and $k = \mathrm{v} = m = 10$, the number of unknowns is somewhere between 3000 and 6600. For $n \geq m$, and $k = \mathrm{v} = m = 20$, the unknowns are between 46,000 and 92,000. Consequently this method, which has recently been studied by BASHARIN (1960), is not applicable for the present purpose, which is to arrive at an analytical explanation of the influence of the interdependence in a link system of a size that usually occurs in practice.

A certain reduction of the number of unknowns can be arrived at if the method of equations of state is limited to describe the traffic conditions for connection from an arbitrary *AB*-column $\mathbf{B}_i$ to an arbitrary outlet column $\mathbf{C}_j$. This method has earlier been studied by the author (1956 and 1957). The weak points in this method are that the influence from the other $k-1$ *B*-columns and the other $\mathrm{v}-1$ outlet columns in the link system must necessarily be approximate and that the number of unknowns is still too large for evaluation of sizes occurring in practice. Since artificial traffic trials on link systems (WALLSTRÖM 1958) have shown that the above-mentioned approximation of the other part of the link system is acceptable, the greatest drawback of the method is that the number of unknowns is too large. The number of unknowns will be 910 for $m = 10$ $(n \geq m)$ and 10,626 for $m = 20$ $(n \geq m)$, which is still too much for numerical calculations.

It is consequently necessary to make further approximations of the general case in order to arrive at numerical results which can be compared with the results of artificial traffic

trials. It is desirable that these approximations be made in such a way that they do not affect the dependence in the link system, which is the main object of this study.

The method described in the sequel is based on a simplification of the equations of state, which makes it simpler to determine the state probabilities for the individual *B*-columns and outlet columns. In spite of this, it will be possible to take into account the dependence between all *B*-columns and all outlet columns. This problem is solved by taking into account that every increase in the number of occupations in a *B*-column, as well as in an outlet column, is dependent on whether it is possible to establish a connection to the desired route or not. The evaluations will require iterations, which are more easily handled by an electronic computer than the former methods in which the large number of memory positions required is difficult or impossible to provide.

2.2 Derivation of the Method

If there were no dependence between the links and the outlets, the traffic distribution in a *B*-column should depend solely on the traffic offered from the inlets in the corresponding *A*-column. So, if the inlets are considered as traffic sources, acting independently of one another, the distribution in the *B*-column should be that of a full availability group. This assumption gives results of satisfactory accuracy for all practical cases with sufficiently large link systems and with low congestion values (Jacobaeus 1950), i.e. when the dependence between *B*-columns and *C*-columns is negligible. However, the purpose of this study is to deal with cases where the dependence is not negligible, as when a link system is used for connection to direct routes in an alternative routing network.

For the traffic distribution in a *B*-column, the dependence in the link system is characterized by the fact that every increase in the number of occupied *B*-links depends on whether any of the free *B*-links can be used for connection to the called outlet column. This fact can be taken into account in the deduction of the distribution for a *B*-column.

Consider an arbitrary *B*-column, $\mathbf{B}_i$, in the two-stage link system shown in *Fig. 1.1.* $[q_i]_t$ denotes the probability of exactly q_i links being occupied at the point of time t.

Here

$$\begin{aligned} &0 \le q_i \le m, \qquad \text{if } n \ge m \\ &0 \le q_i \le n, \qquad \text{if } n \le m \end{aligned}$$

λ_{q_i} denotes the call intensity for successful calls at state q_i. Here

$$\left.\begin{aligned} &\lambda_{q_i} > 0 \text{ for } 0 \le q_i < m, \text{ if } n \ge m, \text{ and} \\ &\qquad \text{for } 0 \le q_i < n, \text{ if } n \le m \\ &\lambda_{q_i} = 0 \text{ for all other } q_i \end{aligned}\right\} \quad (2.2.1)$$

If every inlet in the i:th A-column is assumed to have the call intensity α_i when it is free, λ_{q_i} is written

$$\lambda_{q_i} = (n - q_i) \cdot \alpha_i \cdot R(q_i) \tag{2.2.2}$$

Consequently, λ_{q_i} is dependent on the number of free inlets, $n - q_i$, but also on the factor $R(q_i)$, which is assumed to represent the probability that a new call, when already q_i are present, can be connected to the desired outlet column and can occupy any of the remaining $m - q_i$ free links. Here it must be observed that $(n - q_i) \cdot \alpha_i$ represents the call intensity to the B-column, but $\lambda_{q_i} < (n - q_i) \cdot \alpha_i$ represents the intensity for successful calls.

The durations of the occupations in the B-columns are assumed to be exponentially distributed with the average holding time h, i.e. all calls from the B-columns to all routes have the same holding time distribution. The termination intensity, when q_i links are occupied, is consequently $\mu_{q_i} = \frac{q_i}{h}$, or, if h is taken as time unit,

$$\mu_{q_i} = q_i \tag{2.2.3}$$

The probability of having exactly q_i occupied links at the point of time $t + dt$, dt being infinitesimal, can now be written (cf. JENSEN 1948)

$$[q_i]_{t+dt} = [q_i]_t \cdot \{1 - \lambda_{q_i} \cdot dt - q_i \cdot dt\} + [q_i - 1]_t \cdot \lambda_{q_i - 1} \cdot dt + [q_i + 1]_t \cdot (q_i + 1) \cdot dt + o(dt) \tag{2.2.4}$$

where $\lambda_{q_i} \cdot dt$ is the probability of a successful call and $q_i \cdot dt$ the probability that a call will terminate within the infinitesimal interval $(t + dt)$ when q_i links are occupied.

Assuming statistical equilibrium, i.e. $\frac{d\,[q_i]}{dt} = 0$, we arrive at the following solution of the equation system:

$$[q_i] = \frac{\prod_{\nu_i=0}^{q_i - 1} \lambda_{\nu_i}}{q_i\,!} \cdot [0] \tag{2.2.5}$$

where [0] is defined by

$$\left.\begin{aligned} &\sum_{q_i=0}^{m} [q_i] = 1 \qquad && \text{if } n \geq m \\ &\sum_{q_i=0}^{n} [q_i] = 1 \qquad && \text{if } n \leq m \end{aligned}\right\} \tag{2.2.6}$$

If λ_{ν_i}, defined as in (2.2.2), is inserted in (2.2.5), this expression takes the form

$$[q_i] = \binom{n}{q_i} \cdot \alpha_i^{q_i} \cdot [0] \cdot \prod_{\nu_i = 0}^{q_i - 1} R(\nu_i) \tag{2.2.7}$$

It can be seen that (2.2.7) is the ordinary solution for Erlang's Bernouilli distribution multiplied by the product $\prod_{\nu_i=0}^{q_i-1} R(\nu_i)$. Since these products all have values ≤ 1, the resulting distribution (2.2.7) can be said to be a deformed Erlang's Bernouilli distribution, where the deformation is caused by the dependence between the B-column and the outlet columns in C.

It now remains to give as accurate a description as possible of this deformation. If random hunting is assumed in the link system, the probability $R(q_i)$ can be written in the following way:

$$R(q_i) = 1 - H(m - q_i) \tag{2.2.8}$$

where

$$H(m - q_i) = \sum_{j=1}^{v} P_{ij} \sum_{s_{ij}=0}^{q_i} P(s_{ij} | q_i) \cdot H_j(m - q_i | s_{ij}) \tag{2.2.9}$$

is evidently the probability that a call occurring when $\mathbf{B}_i$ has q_i occupied links will not be successful. In (2.2.9) P_{ij} is the probability that the call is intended for $\mathbf{C}_j$. This probability is the part of all calls from $\mathbf{B}_i$ destined for $\mathbf{C}_j$. It can be written

$$\left.\begin{aligned} &P_{ij} = \eta_{ij} \\ \text{where} \quad &\sum_{j=1}^{v} \eta_{ij} = 1 \end{aligned}\right\} \tag{2.2.10}$$

Expression $P(s_{ij} | q_i)$ is the probability that s_{ij} of the existing q_i occupied links are connected to $\mathbf{C}_j$. This probability can be written

$$P(s_{ij} | q_i) = \binom{q_i}{s_{ij}} \cdot \gamma_{ij}^{s_{ij}} \cdot (1 - \gamma_{ij})^{q_i - s_{ij}} \tag{2.2.11}$$

Since the number s_{ij} depends on the number of successful calls from $\mathbf{B}_i$ to $\mathbf{C}_j$, the value γ_{ij} has the following relation to the quantity η_{ij}, defined in (2.2.10):

$$\left.\begin{aligned} &\gamma_{ij} = K_i \cdot \eta_{ij} \cdot (1 - B_{ij}) \\ &\text{where the constant } K_i \text{ is given a value such that} \\ &\sum_{j=1}^{v} \gamma_{ij} = 1 \end{aligned}\right\} \tag{2.2.12}$$

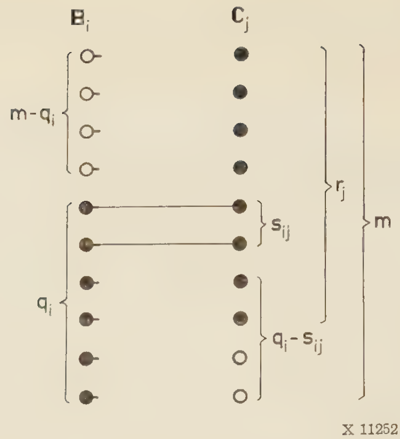

Fig. 2.1. Example of occupations giving congestion from the considered B-column $\mathbf{B}_i$ to the considered outlet column $\mathbf{C}_j$. Blacked-out symbols represent occupied devices.

Here, B_{ij} is the call congestion from $\mathbf{B}_i$ to $\mathbf{C}_j$. B_{ij} is defined by expression (2.2.35) at the end of this section.

The quantity $H_j(m - q \,|\, s_{ij})$ in (2.2.9) is the probability that the $m - q_i$ outlets in $\mathbf{C}_j$ which correspond to the free links in $\mathbf{B}_i$ are all occupied, provided that there are s_{ij} internal connections between the two columns.

For $H_j(m - q_i \,|\, s_{ij})$ the following expression can be derived (cf. *Fig. 2.1*):

$$H_j\,(m - q_i \,|\, s_{ij}) = \sum_{r_j = m - q_i + s_{ij}}^{m} [r_j \,|\, s_{ij}] \cdot \frac{\dbinom{q_i - s_{ij}}{r_j - s_{ij} - m + q_i}}{\dbinom{m - s_{ij}}{r_j - s_{ij}}}$$

or

$$H_j\,(m - q_i \,|\, s_{ij}) = \sum_{r_j = m - q_i + s_{ij}}^{m} [r_j \,|\, s_{ij}] \cdot \frac{\dbinom{q_i - s_{ij}}{m - r_j}}{\dbinom{m - s_{ij}}{r_j - s_{ij}}} \tag{2.2.13}$$

where

$$[r_j \,|\, s_{ij}] = \frac{[r_j]}{\sum\limits_{\varrho_j = s_{ij}}^{m} [\varrho_j]} \tag{2.2.14}$$

Expression (2.2.13) is arrived at by considering the combinatorial probabilities that $r_j - s_{ij}$ occupied outlets are situated in such a way that all outlets, corresponding to free links in $\mathbf{B}_i$, are engaged. The number $r_j - s_{ij}$ is the number of occupations in the outlet column that can be placed independently of the occupations in the considered B-column.

Since the probability $P(s_{ij}|q_i)$ in (2.2.9) already states that there are s_{ij} links connected to the j:th outlet column, the probability of r_j occupations in this column is consequently a conditional probability, which has been written $[r_j|s_{ij}]$. The expression (2.2.14) gives the value of this probability, where r_j can only assume values $\geq s_{ij}$, in accordance with the conditions for r_j, q_i and s_{ij} given in condition (1.1).

It follows from expressions (2.2.7)—(2.2.14) that the distributions $[q_i]$ are dependent on the distributions $[r_j]$ in the outlet columns. For calculating the distributions $[q_i]$, it is consequently necessary to know the state probabilities $[r_j]$ for all outlet columns $\mathbf{C}_j$. In the sequel it will be seen that it is also necessary to know the distributions $[q_i]$ for all $\mathbf{B}_i$ before the distributions $[r_j]$ can be calculated. It follows that an iteration method must be applied for evaluating the distributions $[q_i]$ and $[r_j]$. Further, for calculating an individual case, (ij), in the link system, it is necessary to evaluate the $[q_i]$ for all $\mathbf{B}_i$ and the $[r_j]$ for all $\mathbf{C}_j$.

The state probabilities $[r_j]$, $0 \leq r_j \leq m$, for an arbitrary outlet column, $\mathbf{C}_j$, can be deduced in the same manner as for the state probabilities $[q_i]$. Consequently, analogously to (2.2.5), we have

$$[r_j] = \frac{\prod_{\nu_j=0}^{r_j-1} L_{\nu_j}}{r_j!} \cdot [0] \tag{2.2.15}$$

where

$$\sum_{r_j=0}^{m} [r_j] = 1 \tag{2.2.16}$$

In (2.2.15), L_{ν_j} is the intensity of calls from any of the k inlet columns in stage A of the link system that successfully occupies $\mathbf{C}_j$ when it already has ν_j occupations. The average holding time, h, is assumed to be the same as for the distribution $[q_i]$ and is used as time unit.

To arrive at an expression for L_{ν_j}, it is necessary to define the traffic offered to an outlet column, which is the most difficultly defined quantity in the link system.

There are altogether $n \cdot k$ inlets in the link system, all of which can make calls to the considered outlet column. Consequently, the number of sources for an outlet column can be considered to be at most $n \cdot k$. In every inlet column, i, the call intensity to the j:th outlet

column, $\mathbf{C}_j$, is defined as $\alpha_i \cdot \eta_{ij}$, where α_i is the total call intensity per inlet in A-column i, when the inlet is free, and η_{ij} is the proportion of the calls that are intended for $\mathbf{C}_j$. If there are q_i occupied inlets, $\mathbf{B}_i$ can be said to have the call intensity

$$y_j(q_i) = (n - q_i) \cdot \alpha_i \cdot \eta_{ij} \tag{2.2.17}$$

to the j:th outlet column. At a moment when there are $q_1, q_2 \ldots, q_k$ occcupations in $\mathbf{B}_1$, $\mathbf{B}_2, \ldots, \mathbf{B}_k$, the call intensity for $\mathbf{C}_j$ can be written

$$y_j(q_1, q_2, \ldots, q_k) = \sum_{i=1}^{k} y_j(q_i) = \sum_{i=1}^{k} (n - q_i) \cdot \alpha_i \cdot \eta_{ij} \tag{2.2.18}$$

Here, the call intensity $y_j(q_1, q_2, \ldots, q_k)$ is fully determined by the momentary values of all q_i. To establish a value for L_{r_j} when there are exactly r_j occupations in $\mathbf{C}_j$, it is however necessary to go through all possible combinations of the q_i and combine them with their probabilities. Moreover, to define L_{r_j}, it will be necessary to consider whether the combinations of free and engaged devices in the B-columns and in the outlet columns are such that a call to the route can be successful. Such an expression can in principle be given the form

$$L_{r_j} = \sum_{i} \sum_{q_i} \sum_{s_{ij}} y_j(q_i) \cdot [q_i] \cdot P(s_{ij} | q_i, r_j) \cdot Q(q_i, r_j, s_{ij}) \tag{2.2.19}$$

where the factors have the following meanings:

$y_j(q_i)$ is defined by (2.2.17) and $[q_i]$ by (2.2.7),

$P(s_{ij} | q_i, r_j)$ is the probability that there are s_{ij} connections from $\mathbf{B}_i$ to $\mathbf{C}_j$ when there are q_i and r_j occupations in the two columns,

$Q(q_i, r_j, s_{ij})$ is the probability that the occupations are placed in such a way that a call can be set up from $\mathbf{B}_i$ to $\mathbf{C}_j$.

The summation in (2.2.19) has to be carried out for all possible values of q_i, s_{ij} and i, defined by conditions (1.1) and (1.2). For $P(s_{ij} | q_i, r_j)$ expression (2.2.11) for $P(s_{ij} | q_i)$ can be used with a slight modification, i.e.

$$P(s_{ij} | q_i, r_j) = K \cdot \binom{q_i}{s_{ij}} \cdot \gamma_{ij}^{s_{ij}} \cdot (1 - \gamma_{ij})^{q_i - s_i} \tag{2.2.20}$$

where γ_{ij} is defined in (2.2.12). Here, as given in condition (1.1), s_{ij} can at most be equal to the smallest of q_i and r_j. Therefore, the constant K is given such a value that the sum of the $P(s_{ij} | q_i, r_j)$ for all possible values of s_{ij} is unity.

Finally, for $Q(q_i, r_j, s_{ij})$, the following expression can be used:

$$Q(q_i, r_j, s_{ij}) = 1 - \frac{\binom{q_i - s_{ij}}{m - r_j}}{\binom{m - s_{ij}}{r_j - s_{ij}}} \tag{2.2.21}$$

which is the probability that a call can be set up from $\mathbf{B}_i$ to $\mathbf{C}_j$. The second term in (2.2.21) is the probability that it is not possible. Here, $q_i - s_{ij}$ occupations in $\mathbf{B}_i$, and $r_j - s_{ij}$ occupations in $\mathbf{C}_j$, can be placed on $m - s_{ij}$ possible positions in such a way that all m pairs are blocked. It is here irrelevant which of the m pairs (i, j) are occupied by the s_{ij} internal connections.

The expression (2.2.19) for L_{r_j} is now established with the use of the expressions (2.2.5), (2.2.17), (2.2.20) and (2.2.21). However, this expression is only approximate. The approximation has to do with condition (1.3), which states that the total number of occupations in the B-stage is always equal to the total number of occupations in the outlet columns. From this condition it follows, for example, that if all numbers of occupations, q_i, in the k B-columns and the numbers of occupations, r_j, in $\mathrm{v} - 1$ of the v outlet columns are known, the number of occupations in the remaining outlet column is fully defined. Consequently, the v distributions $[r_j]$ are not independent of one another. The same goes for the k distributions $[q_i]$. The approximation is less severe the larger the link system, i.e. the larger k and v are.

Another drawback of expression (2.2.19) for L_{r_j} is that the summation must be carried out for a large number of combinations of values for q_i and s_{ij}. It is consequently not so well suited for evaluations. There is consequently reason to try to find a simpler approximate expression for L_{r_j}.

Condition (1.3) can be given another form, better suited for calculations, namely

$$\sum_{i=1}^{k} \sum_{q_i=0}^{x} q_i[q_i] = \sum_{j=1}^{\mathrm{v}} \sum_{r_j=0}^{m} r_j[r_j] \tag{2.2.22}$$

$$\begin{aligned} x &= m, && \text{if } n \geq m \\ x &= n, && \text{if } n < m \end{aligned}$$

This condition, which is less severe than (1.3), states that the total traffic carried should on an average be the same in the B-stage and in the outlet columns. The precision of the calculation method will be improved if condition (2.2.22) is used in the evaluations.

4—106690

For L_{r_j}, the following expression will be used instead of (2.2.19):

$$L_{r_j} = A_j \cdot Q(r_j) \tag{2.2.23}$$

Here A_j is a constant for $\mathbf{C}_j$ and $Q(r_j)$ is the probability that a call can successfully occupy an outlet in $\mathbf{C}_j$ when it already has r_j occupations. A_j will be given such a value that condition (2.2.22) is satisfied. Insertion of (2.2.23) in (2.2.15) gives

$$[r_j] = \frac{A_j^{r_j}}{r_j!} \cdot [\mathrm{o}] \cdot \prod_{\nu_j=\mathrm{o}}^{r_j-1} Q(\nu_j) \tag{2.2.24}$$

$$(\mathrm{o} < r_j \leq m)$$

If (2.2.24) is inserted in (2.2.16), the following value for [0] will be obtained:

$$[\mathrm{o}] = \frac{1}{\sum\limits_{r_j=\mathrm{o}}^{m} \frac{A_j^{r_j}}{r_j!} \cdot \prod\limits_{\nu_j=\mathrm{o}}^{r_j-1} Q(\nu_j)} \tag{2.2.25}$$

where

$$\frac{A^{r_j}}{r_j!} \cdot \prod_{\nu_j=\mathrm{o}}^{r_j-1} Q(\nu_j) = 1 \qquad \text{for } r_j = 0$$

From (2.2.24) and (2.2.25) it can be seen that $[r_j]$ is here described as a deformed Erlang distribution, the deformation being represented by the products $\prod Q(\nu_j)$. The advantage of expression (2.2.23) for L_{r_j}, compared with the expression (2.2.19), is that the values of the A_j can be chosen so that condition (2.2.22) is satisfied.

Analogously to the expressions (2.2.8) and (2.2.9), $Q(r_j)$ will be calculated in the following way:

$$Q(r_j) = 1 - H(m - r_j) \tag{2.2.26}$$

where

$$H(m - r_j) = \sum_{i=1}^{k} P_{ij} \sum_{s_{ij}=\mathrm{o}}^{z} P(s_{ij} | r_j) \cdot H_i(m - r_j | s_{ij}) \tag{2.2.27}$$

In the second sum, the summation is carried out to $z =$ the smallest of r_j, m and n.

Since $Q(r_j)$ is the probability that a call can be successfully set up in $\mathbf{C}_j$, $H(m - r_j)$ is the probability that such a trial will fail.

In (2.2.29), P_{ij} is the probability that a call to $\mathbf{C}_j$ will come from $\mathbf{B}_i$.
Here

$$P_{ij} = \Theta_{ij} \tag{2.2.10}$$

is used, where

$$\sum_{i=1}^{k} \Theta_{ij} = 1 \tag{2.2.28}$$

Θ_{ij} is the proportion of all calls offered to $\mathbf{C}_j$ from $\mathbf{B}_i$. For $P(s_{ij}|r_j)$ an expression analogous to (2.2.11) is used:

$$P(s_{ij}|r_j) = \binom{r_j}{s_{ij}} \cdot \delta_{ij}^{s_{ij}} \cdot (1 - \delta_{ij})^{r_j - s_{ij}} \tag{2.2.29}$$

where

$$\delta_{ij} = K_j \cdot \Theta_{ij} \cdot (1 - B_{ij})$$

$$\sum_{i=1}^{k} \delta_{ij} = 1$$

In (2.2.29) δ_{ij} describes the number of successful calls from $\mathbf{B}_i$ to $\mathbf{C}_j$. Here B_{ij} is the call congestion for the case (i, j), defined by (2.2.35) at the end of this section.

Expression (2.2.29) is valid for the case $n \geq m$, and for $r_j \leq n$ in the case $n < m$. For $r_j > n$ in the latter case it follows from condition (1.1),

$$0 \leq s_{ij} \leq q_i \leq n$$

that s_{ij} can only assume values $\leq n$. Consequently, (2.2.29) is here modified into

$$\left.\begin{aligned} P(s_{ij}|r_j) &= \frac{\binom{r_j}{s_{ij}} \cdot \delta_{ij}^{s_{ij}} \cdot (1 - \delta_{ij})^{r_j - s_{ij}}}{\sum\limits_{\sigma=0}^{n} \binom{r_i}{\sigma} \delta_{ij}^{\sigma} \cdot (1 - \delta_{ij})^{r_j - \sigma}} \\ & \qquad\qquad\qquad (0 \leq s_{ij} \leq n)' \\ n < m, & \qquad n < r_j \leq m \end{aligned}\right\} \tag{2.2.29m}$$

$P(s_{ij}|r_j)$ is in this case described as a truncated binomial distribution, which excludes impossible values of s_{ij}.

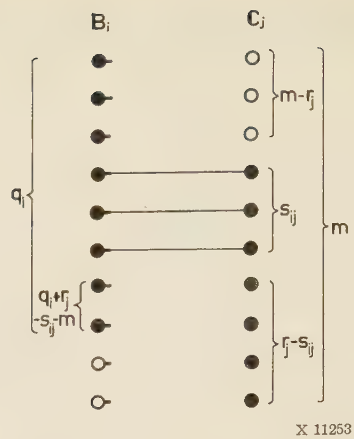

Fig. 2.2. Example of occupations in $\mathbf{B}_i$ and $\mathbf{C}_j$ causing congestion. Blacked-out symbols represent occupied devices.

Finally, the following expression is applied for $H_i(m-r_j|s_{ij})$, (see also *Fig. 2.2*)

$$H_i(m-r_j|s_{ij}) = \sum_{q_i=m-r_j+s_{ij}}^{x} [q_i|s_{ij}] \cdot \frac{\binom{r_j-s_{ij}}{m-q_i}}{\binom{m-s_{ij}}{q_i-s_{ij}}} \tag{2.2.30}$$

$$H_i(m-r_j|s_{ij}) = 0, \quad \text{if} \quad m-r_j+s_{ij} > x$$

where

$$[q_i|s_{ij}] = \frac{[q_i]}{\sum\limits_{\nu_i=s_{ij}}^{x} [\nu_i]} \tag{2.2.31}$$

In (2.2.30) and (2.2.31) the summations for q_i are carried out to the upper limit $q_i = x$, where

$$x = m, \quad \text{if } n \geq m$$
$$x = n, \quad \text{if } n \leq m$$

Analogously to (2.2.13), expression (2.2.30) is arrived at by combinatorial considerations of the possibilities of placing $q_i - s_{ij}$ occupations in $\mathbf{B}_i$ and $r_j - s_{ij}$ occupations in $\mathbf{C}_j$ in such a way that a call cannot be set up from $\mathbf{B}_i$ to $\mathbf{C}_j$. Here, again, the location of the s_{ij} internal

connections is irrelevant. Since it is already stated in (2.2.29) for $P(s_{ij}|r_j)$ that there are s_{ij} such connections, the expression (2.2.31) must be used for $[q_i|s_{ij}]$. This is in accordance with the expression (2.2.14) for $[r_j|s_{ij}]$.

From expressions (2.2.23)—(2.2.31) it is seen that the distributions r_j can now be calculated if the distributions $[q_i]$ are known. For the values of the $[r_j]$ it is furthermore stated that condition (2.2.22) should be fulfilled, which can be done by trying different values for the constants A_j. The method of choosing such values will be described in the next section of this chapter. When the distributions $[q_j]$ and $[r_j]$ are known, it is possible to calculate the distributions $[p_{ij}]$ for the number of engaged pairs, as well as the congestion values.

For $[p_{ij}]$ the following expression is used:

$$[p_{ij}] = \sum_{q_i=0}^{y} [q_i] \sum_{s_{ij}=0}^{q_i} P(s_{ij}|q_i) \cdot P(p_{ij}|q_i, s_{ij}) \tag{2.2.32}$$

(y = the smallest of p_{ij}, m and n.)

Here $[q_i]$ is given in (2.2.7), $P(s_{ij}|q_i)$ in (2.2.11), and $P(p_{ij}|q_i, s_{ij})$ is expressed by

$$P(p_{ij}|q_i, s_{ij}) = \sum_{r_j=p_{ij}-q_i+s_{ij}}^{p_{ij}} \frac{\binom{m-q_i}{p_{ij}-q_i} \cdot \binom{q_i-s_{ij}}{p_{ij}-r_j}}{\binom{m-s_{ij}}{r_j-s_{ij}}} \cdot [r_j|s_{ij}] \tag{2.2.33}$$

In (2.2.33), $[r_j|s_{ij}]$ is given by expression (2.2.14).

For $p_{ij} = m$, the time congestion E_{ij} is obtained. Expression (2.2.32) is then reduced to

$$E_{ij} = \sum_{q_i=0}^{x} [q_i] \sum_{s_{ij}=0}^{q_i} P(s_{ij}|q_i) \cdot H_j(m - q_i|s_{ij}) \tag{2.2.34}$$

$$x = m, \quad \text{if } n \geq m$$
$$x = n, \quad \text{if } n \leq m$$

where $H_j(m - q_i|s_{ij})$ is already defined by (2.2.13) and (2.2.14), i.e.

$$H_j(m - q_i|s_{ij}) = \sum_{r_j=m-q_i+s_{ij}}^{m} \frac{\binom{q_i-s_{ij}}{m-r_j}}{\binom{m-s_{ij}}{r_j-s_{ij}}} \cdot [r_j|s_{ij}] \tag{2.2.13}$$

and

$$[r_j | s_{ij}] = \frac{[r_j]}{\sum\limits_{\varrho_i = s_{ij}}^{m} [\varrho_i]}; \tag{2.2.14}$$

For the call congestion, the following expression is used:

$$B_{ij} = \frac{\sum\limits_{q_i=0}^{x} (n - q_i) \cdot [q_i] \cdot \sum\limits_{s_{ij}=0}^{q_i} P(s_{ij} | q_i) \cdot H_j(m - q_i | s_{ij})}{\sum\limits_{q_i=0}^{x} (n - q_i) \cdot [q_i]} \tag{2.2.35}$$

$$x = m, \quad \text{if } n \geq m$$
$$x = n, \quad \text{if } n \leq m$$

The expected number of calls from $\mathbf{B}_i$ to $\mathbf{C}_j$ is

$$(n - q_i) \cdot \frac{\alpha_i}{1 + \alpha_i \cdot (1 - \overline{B}_i)} \cdot \eta_{ij}$$

when there are q_i occupations in the i:th B-column. Here $\overline{B}_i$ is the average call congestion value for the i:th B-column to all outlet columns (not to be confused with the B-column $\mathbf{B}_i$). Since

$$\frac{\alpha_i}{1 + \alpha_i \cdot (1 - \overline{B}_i)} \cdot \eta_{ij}$$

is independent of q_i, it can be taken outside the summation signs and be abbreviated. Consequently expression (2.2.35) is the ratio of the expected number of blocked calls to the expected number of calls, for the case $\mathbf{B}_i$ to $\mathbf{C}_j$.

2.3 Evaluation of the Method

The intention of this paper is to study whether the analytical description of the dependence phenomena in a link system, as given in the foregoing pages, is sufficiently accurate. This will be done by comparing evaluated values for the distributions in B-columns and outlet columns and for the congestion with results obtained from measurements with artificial traffic on a link system.

Since the evaluations involve an iteration process, it is necessary to determine its starting values. It is also necessary to determine how the trial values in the process shall be changed to arrive at a sufficiently fast convergence.

For a correct comparison between measured and evaluated values, it is essential that the evaluations employ the same assumptions and numerical values as the measurements. It may then be convenient to sum up the assumptions for the artificial traffic measurements (WALLSTRÖM 1961) before continuing to outline the evaluation method.

The measurements are executed on a two-stage link system of the type shown in *Fig. 1.1.* A measurement is made for defined values of m, n, k and v. In the measurements used for this comparison, every inlet in the A-stage is given a call intensity α when the inlet is free. When a call occurs, a random number sequence decides to which outlet column the call is destined. The call intensity per inlet from $\mathbf{B}_i$ to $\mathbf{C}_j$ is consequently $\alpha_i \cdot \eta_{ij}$ when the inlet is free and zero when the inlet is engaged. Here

$$\sum_{j=1}^{\mathrm{v}} \eta_{ij} = 1$$

When the direction of a call has been determined, a test is made whether it is possible to set up the call or not. If it is not possible, the call is rejected. The inlet is freed without any time delay and the call is recorded as rejected. The comparison is made with measurements employing random hunting.

For estimation of the distributions $[q_i]$, $[r_j]$ and $[p_{ij}]$, the number of occupations and the number of occupied pairs are sampled at exponentially distributed time intervals. The accuracy of this sampling method was studied in an earlier paper by WALLSTRÖM (1958).

It follows from the conditions for the measurements that the traffic offered to the different outlet columns is only indirectly defined by the number n and the call intensity values $\alpha_i \cdot \eta_{ij}$. To arrive at comparable results, the evaluation must be based on the same values.

Iteration method

The following principle is used in the iteration process:

1. The distributions $[q_i]$ and $[r_j]$ are evaluated for given starting values.
2. The distributions $[r_j]$ are corrected so that condition (2.2.22) is fulfilled.
3. The distributions $[q_i]$ are re-evaluated for the new distributions $[r_j]$.
4. The distributions $[r_j]$ are corrected so that condition (2.2.22) is fulfilled for the new distributions $[q_i]$.

5. The iteration is continued in this way by alternately correcting the distributions $[q_i]$ and $[r_j]$ until the following three conditions are fulfilled:

$$\left| [q_i]_{(\mu)} - [q_i]_{(\mu-1)} \right| \leq \varepsilon_1 \tag{2.3.1}$$

$$\left| [r_j]_{(\mu)} - [r_j]_{(\mu-1)} \right| \leq \varepsilon_2 \tag{2.3.2}$$

$$\left| \sum_{i=1}^{k} \sum_{q_i=0}^{m \text{ or } n} q_i \, [q_i]_{(\mu)} - \sum_{j=1}^{v} \sum_{r_j=0}^{m} r_j \, [r_j]_{(\mu)} \right| \leq \varepsilon_3 \tag{2.3.3}$$

Here the suffixes (μ) stand for the number of the iteration cycle.

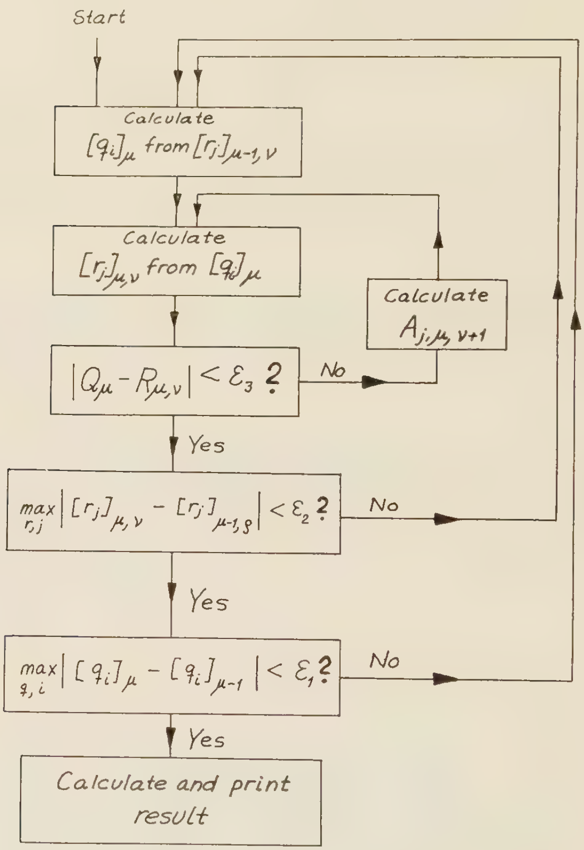

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Fig. 2.3. Iteration principle.

$Q_\mu = \sum \sum q_i \, [q_i]_{(\mu)} \quad R_{\mu,\,\nu} = \sum \sum r_j \, [r_j]_{(\mu,\,\nu)}$

Starting values

For starting the iteration, the distributions $[q_i]$, $(i = 1, 2, \ldots, k)$, are evaluated from the expression

$$\left.\begin{aligned} &[q_i]_{(0)} = \binom{n}{q_i} \cdot \alpha_i^{q_i} \cdot [0]_{(0)} \\ &\sum_{q_i=0}^{x} [q_i]_{(0)} = 1 \\ &x = m, \text{ if } n \geq m \\ &x = n, \text{ if } n \leq m \end{aligned}\right\} \qquad (2.3.4)$$

Here an ordinary Erlang's Bernouilli distribution $(n \geq m)$ or Bernouilli distribution $(n \leq m)$ is used as starting values. This distribution disregards the influence from the outlet columns, as it is taken into account in (2.2.7) by the products $\Pi\, R(\nu)$.

As starting values for the distributions $[r_j]$, $(j = 1, 2, \ldots, \text{v})$, the following values are used for A_j:

$$A_{j(0)} = \sum_{i=1}^{k} n \cdot \frac{\alpha_i}{1 + \alpha_i} \cdot \eta_{ij} \qquad (2.3.5)$$

This expression is too small, since it neglects the fact that congestion here increases the call intensity.

The values for A_j given in (2.3.5) are now used in the first evaluation of the distributions $[r_j]$. Here expression (2.2.24) can be used already in the first evaluation since the distributions $[q_i]_{(0)}$ are now available. The call congestion values B_{ij} given in (2.2.35) are also evaluated. From the B_{ij}-values the average call congestion $\overline{B}_i$ for each B-column is then evaluated from

$$\overline{B}_i = \sum_{j=1}^{\text{v}} \eta_{ij}\, B_{ij} \qquad (2.3.6)$$

A re-evaluation of the A_j is then done with the expression

$$A'_{j(\mu)} = \sum_{i=1}^{k} n \cdot \frac{\alpha_i}{1 + \alpha_i\,(1 - \overline{B}_{i\,(\mu-1)})} \cdot \eta_{ij} \qquad (2.3.7)$$

$$(\mu = 1)$$

If, after this re-evaluation of the $[r_j]$ distributions,

$$R_0 = \sum_{i=1}^{k} \sum_{r_j=1}^{m} r_j[r_j]_{(0)} \tag{2.3.8}$$

is evaluated and compared with

$$Q_0 = \sum_{i=1}^{k} \sum_{q=1}^{n \text{ or } m} q_i[q_i]_{(0)} \tag{2.3.9}$$

a re-evaluation of the $[r_j]$ distribution is made if

$$\left| R_0 - Q_0 \right| > \varepsilon_{30} \tag{2.3.10}$$

New values for A_j are chosen in such a way that

$$\sum_{j=1}^{v} A_{j(\mu)} = \frac{Q_0}{R_0} \cdot \sum_{j=1}^{v} A_{j(\mu-1)} \tag{2.3.11}$$

and the individual $A_{j(\mu)}$ are chosen in the proportion given in (2.3.7).

2.4 Simplified Case

The calculation method given in section 2.2 makes it possible to evaluate link systems in which every B-column and every outlet column have different traffic values. The only obstacle to evaluations for such link systems is the capacity of the computer with respect to evaluation time and memory capacity. The case with completely unsymmetrical loading in a link system has its significance in practice, because the inlet load in a group selector varies from inlet to inlet. Even if the arrangement of the inlets is planned in the best possible way, the traffic per inlet column will vary from day to day and from hour to hour, as can be seen from long-time observations of telephone traffic (see, e.g., ELLDIN, TÅNGE 1960). Consequently the traffic per inlet column, and accordingly per B-column, can never in practice be expected to be the same for all B-columns in a link system. There must always be at least some differences. In the same way the routes in a group selector usually carry traffic flows of varying magnitude. It follows that the unsymmetrical case, treated in section 2.2, is actually a realistic case. The given method can consequently, if such subtle methods are necessary, be used for studying the effect of unsymmetrical loading in a link system. On the other hand, it is not necessary to apply the method to so complicated a case for explaining the interdependence effect in a link system. Here some simplifications can be made without upsetting the explanation given in the foregoing pages.

The first way of reducing the evaluation work is to assume that all inlets in the link system have the same call intensity,

$$\alpha_i = \alpha \tag{2.4.1}$$

$$(i = 1, 2, \ldots, k)$$

and that all inlet columns have the same call intensity to the same route, i.e.

$$\eta_{ij} = \eta_j \tag{2.4.2}$$

$$(i = 1, 2, \ldots, k \text{ and } j = 1, 2, \ldots, \mathrm{v})$$

while the outlet columns may be offered different traffic flow quantities. Here all B-columns have the same traffic distributions

$$[q_i] = [q] \tag{2.4.3}$$

$$(i = 1, 2, \ldots, k)$$

This simplification renders all summations for different B-columns in the formulae unnecessary and shortens the evaluations. The interdependence is unaffected.

Analogously, the evaluations are reduced if similar assumptions are made for the outlet columns. Here the traffic offered to the columns can either be assumed to be the same for all columns or to take only a very limited number of different values. For the case of all routes being offered the same amount of traffic, we have

$$\eta_{ij} = \eta = \frac{1}{\mathrm{v}} \tag{2.4.4}$$

$$(i = 1, 2, \ldots, k \qquad j = 1, 2, \ldots, \mathrm{v})$$

If, for instance, only two types of outlet columns are assumed, we have

$$\begin{aligned} \eta_{ij} &= \eta_{\mathrm{I}} \\ \eta_{ij} &= \eta_{\mathrm{II}} \\ (i &= 1, 2, \ldots, k) \end{aligned} \tag{2.4.5}$$

Here the first equation is valid for outlet columns j belonging to the first type, and the second equation for j:s belonging to the second type.

Simplified Case II

The following simplifications are made in the expressions given in section 2.2 for the case of all B-columns being equal and the v outlet columns being of only two types with respect to the traffic offered to them. Here $\mathrm{v_I}$ outlet columns are assumed to be of the first type and $\mathrm{v_{II}}$ of the second type:

$$\mathrm{v_I} + \mathrm{v_{II}} = \mathrm{v} \tag{2.4.6}$$

The case will be called Simplified Case II, where II refers to the number of different types of outlet columns. In chapter 3 it will be abbreviated HS II.

As follows from (2.4.1) and (2.4.3), expression (2.2.7) can be written

$$[q] = \binom{n}{q} \cdot \alpha^q \cdot [0] \cdot \prod_{v=0}^{q-1} R(v) \tag{2.2.7a}$$

where

$$\sum_{q=0}^{x} [q] = 1$$

$$(x = m \text{ if } n \geq m \text{ and } x = n \text{ if } n < m)$$

Then

$$R(q) = 1 - H(m-q) \tag{2.2.8a}$$

and

$$H(m-q) = \sum_{j=\mathrm{I}}^{\mathrm{II}} P_j \sum_{s_j=0}^{q} P(s_j|q) \cdot H_j(m-q|s_j) \tag{2.2.9a}$$

Here

$$\begin{aligned} P_{\mathrm{I}} &= \mathrm{v}_{\mathrm{I}} \cdot \eta_{\mathrm{I}} \\ P_{\mathrm{II}} &= \mathrm{v}_{\mathrm{II}} \cdot \eta_{\mathrm{II}} \\ P_{\mathrm{I}} + P_{\mathrm{II}} &= 1 \end{aligned} \tag{2.2.10a}$$

where η_{I} and η_{II} are defined in (2.4.5).

For $P(s_j|q)$ we have

$$P(s_j|q) = \binom{q}{s_j} \cdot \gamma_j^{s_j} \cdot (1-\gamma_j)^{q-s} \qquad (j = \mathrm{I} \text{ or } \mathrm{II}) \tag{2.2.11a}$$

Here γ_j is defined by

$$\begin{aligned} &\gamma_j = K \cdot \eta_j \cdot (1 - B_j) \\ &(j = \mathrm{I} \text{ or } \mathrm{II}) \\ &\mathrm{v}_{\mathrm{I}} \cdot \gamma_{\mathrm{I}} + \mathrm{v}_{\mathrm{II}} \cdot \gamma_{\mathrm{II}} = 1 \end{aligned} \tag{2.2.12a}$$

where B_j is the call congestion to the outlet column of type j.

For $H_j(m-q|s_j)$ we have the following expression

$$H_j(m-q|s_j) = \sum_{r=m-q+s_j}^{m} [r_j|s_j] \cdot \frac{\binom{q-s_j}{m-r_j}}{\binom{m-s_j}{r_j-s_j}} \tag{2.2.13a}$$

where

$$[r_j|s_j] = \frac{[r_j]}{\sum\limits_{\varrho_j = s_j}^{m} [\varrho_j]} \tag{2.2.14a}$$

For the distributions in the two types of outlet columns, the following modified expressions can now be used:

$$[r_j] = \frac{A_j^{r_j}}{r_j!} \cdot [0] \cdot \prod_{\nu_j = 0}^{r_j - 1} Q[\nu_j] \tag{2.2.24a}$$

$$(j = \mathrm{I} \text{ or } \mathrm{II})$$

where A_j is given such a value that the equation

$$k \sum_q q[q] = \sum_{j=\mathrm{I}}^{\mathrm{II}} v_j \sum_{r_j=0}^{m} r_j[r_j] \tag{2.2.22a}$$

is satisfied.

In (2.2.24a) the $Q(\nu_j)$ are defined by the expressions

$$Q(r_j) = 1 - H(m - r_j) \tag{2.2.26a}$$

$$H(m - r_j) = \sum_{s_j=0}^{r_j} P(s_j|r_j) \cdot H_i(m - r_j|s_j) \tag{2.2.27a}$$

In (2.2.27a), $P(s_j|r_j)$ is expressed as

$$\left.\begin{aligned} & P(s_j|r_j) = \binom{r_j}{s_j} \cdot \delta_j^{s_j} \cdot (1 - \delta_j)^{r_j - s_j} \\ & \text{where} \quad \delta_j = \frac{1}{k} \\ & \qquad \text{for } n \geq m \text{ or } n < m \text{ and } r_j < n \end{aligned}\right\} \tag{2.2.29a}$$

and

$$\left.\begin{aligned} & P(s_j|r_j) = \frac{\binom{r_j}{s_j} \cdot \delta_j^{s_j} \cdot (1 - \delta_j)^{r_j - s_j}}{\sum\limits_{\sigma_j=0}^{n} \binom{r_j}{\sigma_j} \cdot \delta_j^{\sigma_j} \cdot (1 - \delta_j)^{r_j - \sigma_j}} \\ & \text{for } n < m \text{ and } n < r_j \leq m \end{aligned}\right\} \tag{2.2.29ma}$$

Here it is equally probable that a successful call will originate from any of the k B-columns.

For $H_i(m - r_j | s_j)$ in (2.2.27a), the following expression is used:

$$H_i(m - r_j | s_j) = \sum_{q=m-r_j+s_j}^{x} [q | s_j] \cdot \frac{\binom{r_j - s_j}{m - q}}{\binom{m - s_j}{q - s_j}} \tag{2.2.30a}$$

where

$$[q | s_j] = \frac{[q]}{\sum_{v=s_j}^{x} [v]} \tag{2.2.31a}$$

In (2.2.30a) and (2.2.31a), where the only change is that the suffixes i have been taken away, the summation limit x has the same significance as before, viz.

$x = m$, if $n \geq m$

and

$x = n$, if $n \leq m$

For the distributions $[p_{ij}]$, the number of occupied pairs, only a slight modification is made:

$$[p_j] = \sum_{q=0}^{y} [q] \sum_{s_j=0}^{q} P(s_j | q) \cdot P(p_j | q, s_j) \tag{2.2.32a}$$

(y = the smallest of p_j, n and m.)

Here, $[q]$ is given by (2.2.7a), $P(s_j | q)$ by (2.2.11a), and $P(p_j | q, s_j)$ by

$$P(p_j | q, s_j) = \sum_{r_j=p_j-q+s_j}^{p_j} \frac{\binom{m - q}{p_j - q}\binom{q - s_j}{p_j - r_j}}{\binom{m - s_j}{r_j - s_j}} \cdot [r_j | s_j] \tag{2.2.33a}$$

$[r_j | s_j]$ being defined in (2.2.14a).

For $p_j = m$, the time congestion E_j is written

$$E_j = \sum_{q=0}^{x} [q] \sum_{s_j=0}^{q} P(s_j | q) \cdot H_j(m - q | s_j) \tag{2.2.34a}$$

($x = m$ if $n \geq m$ and $x = n$ if $n \leq m$)

$H_j(m - q | s_j)$ being given in (2.2.13a).

The call congestion is in this case

$$B_j = \frac{\sum_{q=0}^{x} (n-q) \cdot [q] \sum_{s_j=0}^{q} P(s_j|q) \cdot H_j(m-q|s_j)}{\sum_{q=0}^{x} (n-q) \cdot [q]} \tag{2.2.35a}$$

all quantities in (2.2.35a) being defined above.

The evaluations are carried out on the principles given in section 2.3 and are considered as finished when conditions (2.3.1), (2.3.2) and (2.3.3) are fulfilled.

Simplified Case I

The simplest possible case is arrived at when all outlet columns as well are assumed to be offered the same traffic flow from every B-column. This case will be referred to as HS I in the comparisons in chapter 3.

For the fully symmetrical case the following relations are valid:

$$\alpha_i = \alpha \tag{2.4.1}$$

$$\eta_{ij} = \eta = \frac{1}{\mathrm{v}} \tag{2.4.7}$$

$$\Theta_{ij} = \Theta = \frac{1}{k} \tag{2.4.8}$$

$$[q_i] = [q] \tag{2.4.3}$$

$$[r_j] = [r] \tag{2.4.9}$$

$$[p_{ij}] = [p] \tag{2.4.10}$$

$$E_{ij} = E \tag{2.4.11}$$

$$B_{ij} = B \tag{2.4.12}$$

$(i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, \mathrm{v})$

For the evaluation the following expressions are used:

$$[q] = \binom{n}{q} \cdot \alpha^q \cdot [0] \cdot \prod_{\nu=0}^{q-1} R(\nu) \tag{2.2.7b}$$

(identical to (2.2.7a))

$$R(q) = 1 - H(m-q) \tag{2.2.8b}$$

(identical to (2.2.8a))

$$H(m-q) = \sum_{s=0}^{q} P(s|q) \cdot H_j(m-q|s) \tag{2.2.9b}$$

where

$$P(s|q) = \binom{q}{s} \cdot \eta^s \cdot (1-\eta)^{q-s} \tag{2.2.11b}$$

η being defined in (2.4.7)

$$H_j(m-q|s) = \sum_{r=m-q+s}^{m} [r|s] \frac{\binom{q-s}{r-s}}{\binom{m-s}{r-s}} \tag{2.2.13b}$$

and

$$[r|s] = \frac{[r]}{\sum\limits_{\varrho=s}^{m} [\varrho]} \tag{2.2.14b}$$

Further, we have

$$[r] = \frac{A^r}{r!} \cdot [0] \cdot \prod_{\varrho=0}^{r-1} Q(\varrho) \tag{2.2.24b}$$

where A is determined so as to satisfy

$$k \cdot \sum_{q=1}^{x} q \cdot [q] = \mathrm{v} \cdot \sum_{r=1}^{m} r \cdot [r] \tag{2.2.22b}$$

($x = m$, if $n \geq m$ and $x = n$, if $n \leq m$).

Further

$$Q(r) = 1 - H(m-r) \tag{2.2.26b}$$

$$H(m-r) = \sum_{s=0}^{r} P(s|r) \cdot H_i(m-r|s) \tag{2.2.27b}$$

$$P(s|r) = \binom{r}{s} \cdot \Theta^s \cdot (1-\Theta)^{r-s} \tag{2.2.29b}$$

Θ being defined in (2.4.8)

$$H_i(m-r|s) = \sum_{q=m-r+s}^{x} [q|s] \cdot \frac{\binom{r-s}{m-q}}{\binom{m-s}{q-s}} \tag{2.2.30b}$$

$$[q|s] = \frac{[q]}{\sum\limits_{v=s}^{x} [v]} \qquad (x \text{ is defined above}) \tag{2.2.31b}$$

The distribution for the number of occupied pairs is now

$$[p] = \sum_{q=0}^{y} [q] \sum_{s=0}^{q} P(s|q) \cdot P(p|q, s) \tag{2.2.32b}$$

(y = the smallest of p, m and n)

The expression $[q]$ is given in (2.2.7b), $P(s|q)$ in (2.2.11b), and

$$P(p|q, s) = \sum_{r=p-q+s}^{p} \frac{\binom{m-q}{p-q} \cdot \binom{q-s}{p-r}}{\binom{m-s}{r-s}} \cdot [r|s] \tag{2.2.33 b}$$

$[r|s]$ being defined in (2.2.14b).

For $p = m$, the time congestion from any B-column to any outlet column is written

$$E = \sum_{q=0}^{x} [q] \cdot H(m-q) \tag{2.2.34b}$$

where $H(m-q)$ is defined by (2.2.9b) and x has the same meaning as above.

Expression (2.2.34b) is formally in conformity with the original expression for the congestion given by JACOBAEUS (1950). The improvement arrived at here is that the number of possible connections between the considered B-column and the considered outlet column has been taken into account, as well as the deformation of the distributions $[q]$ and $[r]$.

Analogously to (2.2.34b) the time congestion can also be written

$$E = \sum_{r=0}^{m} [r] \cdot H(m-r) \tag{2.4.13}$$

where $H(m-r)$ is defined by (2.2.27b).

5—106690

Finally, the call congestion is written

$$B = \frac{\sum_{q=0}^{x} (n-q) \cdot [q] \cdot H(m-q)}{\sum_{q=0}^{x} (n-q) \cdot [q]} \tag{2.2.35b}$$

The evaluation is carried out on the same lines as for the iteration process in section 2.3 and is considered as finished when conditions (2.3.1), (2.3.2) and (2.3.3) are fulfilled.

2.5 Modified Method

As can be seen from section 2.3, the length of the iteration process is determined by the three conditions

$$\left| [q_i]_{(\mu)} - [q_i]_{(\mu-1)} \right| \leq \varepsilon_1 \tag{2.3.1}$$

$$\left| [r_j]_{(\mu)} - [r_j]_{(\mu-1)} \right| \leq \varepsilon_2 \tag{2.3.2}$$

$$\left| Q_{(\mu)} - R_{(\mu)} \right| \leq \varepsilon_3 \tag{2.3.3}$$

The first two conditions check that the iteration has gone so far that the deformation effect is satisfactorily described. The third condition checks that the average number of occupied devices is sufficiently similar in the B-stage and in the outlets or, in other words, that the values A_j in expression (2.2.24) for $[r_j]$ are chosen in conformity with condition (2.3.3). As can be seen from item 2 of the description of the iteration method in section 2.3, condition (2.3.3) means that the distributions $[r_j]$ may be re-evaluated several times before the next evaluation of the $[q_i]$ can start. Consequently, if the correct values of A_j can be chosen from the beginning, the iteration process can be shortened.

In order to arrive at a direct determination of the correct values for A_j, Simplified Case I is used for the study. Here all B-columns offer the same traffic to all outlet columns, as stated in the conditions (2.4.1)—(2.4.12).

For determination of the traffic offered to an outlet column, it is convenient to study first the expression for the traffic carried by an outlet column. We have

$$A'_c = \sum_{r=1}^{m} r \cdot [r] \tag{2.5.1}$$

From (2.2.24) it follows that

$$r \cdot [r] = A \cdot Q(r-1) \cdot [r-1] \tag{2.5.2}$$

Insertion of (2.5.2) in (2.5.1) gives

$$A_c' = \sum_{r=0}^{m} A \cdot Q(r) \cdot [r] \tag{2.5.3}$$

where $Q(m) = 0$.

$Q(r)$ expresses the probability that a call can successfully be set up from a B-column to the considered outlet column when r outlets are already occupied in it.

From (2.2.26) and (2.4.13) it follows that (2.5.3) can be written

$$A_c' = A \cdot (1 - E) \tag{2.5.4}$$

because

$$\sum_{r=0}^{m} [r] \cdot Q(r) = 1 - E \tag{2.5.5}$$

Hence the traffic offered to the outlet column is

$$A_c = \frac{A_c'}{1 - E} \tag{2.5.6}$$

where A_c is identical to the constant A in expression (2.2.24b) for $[r]$, i.e.

$$A_c = A$$

If the link system had no internal congestion, all $Q(r)$ would be unity for $0 \leq r < m$ and the expression (2.2.24b) would be reduced to a pure Erlang distribution

$$[r] = \frac{A^r}{r!} \cdot [0]$$

For this distribution the traffic offered is A. Consequently a pure Erlang distribution and an Erlang distribution deformed in the way assumed for expression (2.2.24b) both have the same value for the traffic offered. Since the time congestion is different in the two cases, however, the traffic carried also differs.

The call congestion, B, is defined by expression (2.2.35b). If $H(m - r)$ is substituted by $1 - R(q)$, B can be written

$$B = \frac{\sum_{q=0}^{x} (n - q) \cdot \alpha \cdot [q] \cdot \{1 - R(q)\}}{\sum_{q=0}^{x} (n - q) \cdot \alpha \cdot [q]} \tag{2.5.7}$$

where, as before,

$$x = m, \quad \text{if } n \geq m$$

$$x = n, \quad \text{if } n \leq m$$

and

$$R(m) = 0$$

If we define the traffic offered to a B-column as

$$A_B = \sum_{q=0}^{x} (n - q) \cdot \alpha \cdot [q] \tag{2.5.8}$$

expression (2.5.7) can be written

$$B = \frac{A_B - A'_B}{A_B} \tag{2.5.9}$$

where

$$A'_B = \sum_{q=0}^{x} (n - q) \cdot \alpha \cdot [q] \cdot R(q) \tag{2.5.10}$$

is the traffic carried by the B-column.

Expression (2.5.9) can also be written

$$A_B = \frac{A'_B}{1 - B} \tag{2.5.11}$$

It is significant that the expression for the traffic offered to the B-column, A_B, contains the call congestion B, while the corresponding expression for the traffic offered to the outlet column, (2.5.6), depends on the time congestion E in the link system.

If expression (2.5.8) is split into two sums, we obtain

$$A_B = n \cdot \alpha \sum_{q=0}^{x} [q] \quad - \quad \alpha \cdot \sum_{q=0}^{x} q[q]$$

where

$$\sum_{q=0}^{x} [q] = 1$$

and

$$\sum_{q=0}^{x} q \cdot [q] = A'_B = A_B \cdot (1 - B)$$

and consequently

$$A_B = n \cdot \alpha - \alpha \cdot A_B \cdot (1 - B)$$

$$A_B = \frac{n \cdot \alpha}{1 + \alpha \cdot (1 - B)} \tag{2.5.12}$$

$$A'_B = \frac{n \cdot \alpha \cdot (1 - B)}{1 + \alpha \cdot (1 - B)} \tag{2.5.13}$$

These expressions (2.5.12) and (2.5.13) are formally identical to the expressions given by PALM (1946) and COHEN (1958) for a full availability group with m devices and with a limited number of traffic sources, n, when $n > m$. The only difference is that B in the above expressions is the call congestion in the link system and not that of a fully available group.

Using the condition

$$k \cdot \sum_{q=0}^{x} q[q] = \mathrm{v} \cdot \sum_{r=0}^{m} r[r] \tag{2.2.22b}$$

we have

$$k \cdot A'_B = \mathrm{v} \cdot A'_C \tag{2.5.14}$$

Introducing (2.5.4), (2.5.11) and (2.5.13) in the above expression we obtain

$$A = \frac{k}{\mathrm{v}} \cdot \frac{1 - B}{1 - E} \cdot \frac{n \cdot \alpha}{1 + \alpha \cdot (1 - B)} \tag{2.5.15}$$

It is consequently possible to express the traffic offered to an outlet column if the time congestion E and the call congestion B are known.

From expressions (2.5.4) and (2.5.14) the following expression for A can also be deduced:

$$A = \frac{k}{\mathrm{v}} \cdot \frac{A'_B}{1 - E} = \frac{k}{\mathrm{v}} \cdot \frac{\sum_{q=0}^{x} q[q]}{1 - E} \tag{2.5.16}$$

This expression could have been used in the evaluations for Simplified Case I, i.e.

$$A_{(\mu)} = \frac{k}{\mathrm{v}} \cdot \frac{\sum_{q=0}^{x} q[q]_{(\mu - 1)}}{1 - E_{(\mu - 1)}} \tag{2.5.17}$$

where the μ:th improved trial value for A in the iteration is calculated from values of $[q]$ and E in the preceding cycle No. $\mu - 1$. The expression has, however, no advantage over the expression (2.3.11b) which was recommended for the case.

The expressions (2.5.15)—(2.5.17) show, however, that the traffic offered to an outlet column cannot be determined without knowing in advance the resulting congestion. It follows that it is not possible to shorten the iteration process by giving A a correct value from the beginning. The only way to reduce the evaluation work is to modify the assumptions on which the calculations were based in the previous pages.

Since the evaluation results are to be compared with measured results, one way is to use measured values of the call and time congestion in the expression (2.5.15), i.e.

$$A = \frac{k}{v} \cdot \frac{1 - B_M}{1 - E_M} \cdot \frac{n \cdot \alpha}{1 + \alpha \cdot (1 - B_M)} \tag{2.5.18}$$

where the suffix M indicates values taken from measurements. Here condition (2.2.22b) is only satisfied if the evaluation gives the same values for B and E as the measurements. In principle this can never be expected to occur, since the calculation method is approximate. The use all through the evaluation of expression (2.5.18) as a constant value for A can consequently be considered as a modification of the method described in the previous sections, since condition (2.2.22b) is excluded. The method will hereafter be referred to as the Modified Method (abbreviated HM in chapter 3). Since condition (2.2.22b) is not fulfilled, it can be expected to have less precision than the other method. It may however tell us something about the significance of the different conditions for the link system and it may be desirable to compare the results of this method with those of the previous one. In fact, this modified method was first tried in 1959 by the author and later on modified to the one described in sections 2.2—2.4.

For the evaluation of the modified method the following scheme is followed:

$$[q] = \binom{n}{q} \cdot \alpha^q \cdot [0] \cdot \prod_{\nu=0}^{q-1} R(\nu) \tag{2.2.7c}$$

where

$$R(q) = 1 - H(m - q) \tag{2.2.8c}$$

and

$$H(m - q) = \sum_{s=0}^{q} P(s|q) H_j(m - q|s) \tag{2.2.9c}$$

$$P(s|q) = \binom{q}{s} \cdot \eta^s \cdot (1 - \eta)^{q-s} \tag{2.2.11c}$$

(η is defined in 2.4.7)

$$H_j(m - q|s) = \sum_{r=m-q+s}^{m} [r|s] \cdot \frac{\binom{q - s}{m - r}}{\binom{m - s}{r - s}} \tag{2.2.13c}$$

$$[r|s] = \frac{[r]}{\sum_{\varrho=s}^{m} [\varrho]} \tag{2.2.14c}$$

In the first evaluation of $[q]$, we use

$$[r] = \frac{A^r}{r!} \cdot [0] \tag{2.5.19c}$$

in (2.2.14c), where A is given in (2.5.18), which expression uses measured values for B and E. In the following iterations the latest evaluation of $[r]$ is used.

When $[q]$ has been determined, the distribution $[r]$ is evaluated from

$$[r] = \frac{A^r}{r!} \cdot [0] \cdot \prod_{\varrho=0}^{r-1} Q(\varrho) \tag{2.2.24c}$$

where A is given in (2.5.18), and

$$Q(r) = 1 - H(m-r) \tag{2.2.26c}$$

$$H(m-r) = \sum_{s=0}^{r} P(s|r) \cdot H_i(m-r|s) \tag{2.2.27c}$$

$$P(s|r) = \binom{r}{s} \cdot \Theta^s \cdot (1-\Theta)^{r-s} \tag{2.2.29c}$$

$$\Theta = \frac{1}{k} \tag{2.4.8}$$

$$H_i(m-r|s) = \sum_{q=m-r+s}^{x} [q|s] \frac{\binom{r-s}{m-q}}{\binom{m-s}{q-s}} \tag{2.2.30c}$$

$$[q|s] = \frac{[q]}{\sum_{\nu=s}^{x} [\nu]} \tag{2.2.31c}$$

($x = m$ if $n \geq m$, and $x = n$ if $n \leq m$).

The above expressions are identical to those given for $[q]$ and $[r]$ in Simplified Case I, section 2.4.

The evaluations are carried on until the conditions

$$\left| [q]_{(\mu)} - [q]_{(\mu-1)} \right| \leq \varepsilon_1 \tag{2.3.1c}$$

$$\left| [r]_{(\mu)} - [r]_{(\mu-1)} \right| \leq \varepsilon_2 \tag{2.3.2c}$$

are satisfied. The distribution $[p]$ for the number of occupied pairs is then evaluated from

$$[p] = \sum_{q=0}^{y} [q] \cdot \sum_{s=0}^{q} P(s|q) \cdot P(p|q, s) \tag{2.2.32c}$$

where y is the smallest of p, m and n, and $P(s|q)$ is defined by (2.2.11c) and finally

$$P(p|q, s) = \sum_{r=p-q+s}^{p} \frac{\binom{m-q}{p-q}\binom{q-s}{p-r}}{\binom{m-s}{r-s}} \cdot [r|s] \tag{2.2.33c}$$

$[r|s]$ being defined in (2.2.14c).

For $p = m$, the time congestion is evaluated from

$$E = \sum_{q=0}^{x} [q] \cdot H(m-q) \tag{2.2.34c}$$

x being defined above and $H(m-q)$ in (2.2.9c).

The call congestion is evaluated from

$$B = \frac{\sum_{q=0}^{x} (n-q) \cdot [q] \cdot H(m-q)}{\sum_{q=0}^{x} (n-q) \cdot [q]} \tag{2.2.35c}$$

In the evaluations use has not been made of the condition (2.3.3) which states that the traffic carried in the B-columns and in the outlet columns should be approximately equal. Here this condition will be fulfilled only when measured and evaluated values for B and E are identical. Since the measured values in themselves always have a certain degree of inaccuracy, and since the calculation method is approximate, condition (2.3.3) cannot be expected to be fulfilled. From this fact it follows that the Modified Method described in this section can be expected to be of less precision than the former method.

Comparison with Results of Artificial Traffic Measurements

A comparison will now be made between calculated and measured values for the distributions [p], [q] and [r] and for time and call congestion, E and B. The theoretical values, as given in the preceding chapter, were evaluated in the Swedish electronic computer BESK. The programming was done by Mr. S-G. Carlsson, Telefonaktiebolaget L M Ericsson. The measurements used for this comparison are part of the series of artificial traffic measurements on link systems presented in this issue of Ericsson Technics (WALLSTRÖM 1961).

The following hypotheses are compared with measured values:

Hypothesis		
HE	Equations of State Method	(ELLDIN 1956, 1957)
HS I	Simplified Case I	(See sect. 2.4)
HS II	Simplified Case II	(See sect. 2.4)
HM	Modified Case	(See sect. 2.5)
HI	Independence Assumption	(See the following)

Two types of measurements are used for the comparisons, corresponding to the conditions for HS I and HS II. Hence, in the first case, *measurements of type 1*, the link system is symmetrically loaded. Every B-column is offered the same traffic and every B-column offers the same traffic to all outlet columns. In the second case, *measurements type 2*, every B-column is offered the same traffic, but offers different traffic quantities to two types of outlet columns. In both cases an outlet column is considered as a separate route. There are no overflow routes in the link system, and blocked calls are immediately rejected and the calling inlet source is immediately freed.

Comparison No. 1.

In the first step of comparison, the precision of the new method, HS, is compared with that of the equations of state method, HE, used in earlier studies. This comparison is made for a relatively small link system ($m = 4$) and for measurements of type 1. Measured values are also compared with the hypotheses HM and HI.

Comparison No. 2.

For a link system with $m = 4$, hypotheses HS II and HI are compared with measurements of type 2. This case is of less practical interest since the link system is small, but may give some items of information as regards the precision of HS II. The case was originally used

for checking the evaluation programme for HS I and HS II, since it was possible to make a manual evaluation with a desk calculator for this case and compare the result with the automatic evaluation. Here it was found that the electronic computer was about 8000 times as fast as a first class human evaluator.

Comparison No. 3

The third step of comparison is for a link system with $m = 10$, where hypotheses HS I, HM and HI are compared with measurements of type 1. Here the comparison is made for a link system of a size occurring in practice.

Comparison No. 4

Finally, hypotheses HS II and HI are compared with some results of measurements type 2 for a link system with $m = 10$.

Measures for the comparisons

For estimating the accuracy of the calculated values compared with the exact ones, represented by measured values having a certain degree of inaccuracy, the precision of the measured values must be defined. This is easily done without much theoretical consideration, since the result of a measurement is divided into five parts and the results of these individual parts can be used for estimation of the accuracy, here best represented by the standard error of the average values for the whole measurement. Following this method, given by Wallström (Wallström 1958, Section 4.1, p. 275, eq. (4.1.1)), the standard error for a measured value P^* will be

$$dP^* = \sqrt{\frac{1}{N(N-1)} \cdot \sum_{i=1}^{n} (P_i^* - P^*)^2} \tag{3.1}$$

where

$$P^* = \frac{1}{N} \sum_{i=1}^{n} P_i^*$$

is the average for the N parts of the measurement and P_i^* the individual result for the i:th part.

Since it is already known that the hypotheses are approximate, expression (3.1) could be used for judging whether the measurement has sufficient precision to enable the systematical deviation of an approximate value to be distinguished from an exact one. Following this line of thought it can also be estimated how long a period a measurement should cover to give a significant deviation. An estimate of the accuracy of an approximation can then be

expressed as the time it takes before a deviation can be expected to be significant with a certain reliability. Such an estimation will not be made here.

For comparing measured and calculated values of the distributions, the differences

$$\Delta_\nu = [\nu]_H - [\nu]_M \tag{3.2}$$

$$(\nu = p,\ q \text{ or } r)$$

between hypothetical (H) and measured (M) values are calculated. As a simple concluding measure

$$\sum |\Delta_\nu| = \sum_{\nu=0}^{m} |[\nu]_H - [\nu]_M| \tag{3.3}$$

is used.

As a supplementary measure for the divergence between calculated and measured values

$$\frac{\chi^2}{N} = \sum_{\nu=0}^{m} \frac{([\nu]_M - [\nu]_H)^2}{[\nu]_H} = \sum_{\nu=0}^{m} \frac{\Delta_\nu^2}{[\nu]_H} \tag{3.4}$$

is used. The measure (3.4) refers to a χ^2-distribution with N independent observations. Since the sampling of the number of occupations ν for determination of $[\nu]_M$ is made at exponentially distributed intervals, every individual observation result is more or less dependent on the preceding one. The number of independent observations is consequently not directly given from the measurement, but is a hypothetical number which has to be estimated in one way or another. However, since the requirement is not to reject hypotheses but only to compare a number of them with the same measurement result, it is unnecessary to determine the hypothetical number of independent observations, N. Consequently, the deviation measure $\frac{\chi^2}{N}$ is sufficient.

For the comparison between calculated and measured values for the time and call congestion, E and B, the differences are defined as

$$\left.\begin{aligned} \Delta_E &= E_H - E_M \\ \Delta_B &= B_H - B_M \end{aligned}\right\} \tag{3.5}$$

and the relative differences as

$$\left.\begin{aligned} \delta_E &= \frac{\Delta_E}{E_M} = \frac{E_H - E_M}{E_M} \\ \delta_B &= \frac{\Delta_B}{B_M} = \frac{B_H - B_M}{B_M} \end{aligned}\right\} \tag{3.5a}$$

To determine whether the measurements are accurate enough to distinguish between calculated values and measured values,

$$\left.\begin{aligned} \lambda_E &= \frac{\Delta_E}{dE_M} = \frac{E_H - E_M}{dE_M} \\ \lambda_B &= \frac{\Delta_B}{dB_M} = \frac{B_H - B_M}{dB_M} \end{aligned}\right\} \quad (3.6)$$

will be calculated, where dE_M and dB_M are determined as given in (3.1).

Hypothesis HI

In order to obtain a survey of the improvements arrived at by taking into account the dependence in the link system, the distributions and congestion values are also calculated under the assumption of independence between the two parts of the link system. These calculations, which are done by ordinary methods, are in the comparisons called hypothesis *HI*.

For the evaluations, the following value for the traffic offered to a route is used

$$A_i = \frac{\sum_{r_i=0}^{m} r_i[r_i]_M}{1 - E_{i,\,M}} \tag{3.7}$$

where suffix i can be omitted for measurements of type 1 and takes the values $i = \text{I}$ and II for measurements of type 2. The expression (3.7) gives the traffic flow actually offered to the route in the measurements.

For evaluating the traffic carried in the *B*-columns, the values α for the measurements are used. Hence, the traffic carried per inlet device is calculated as

$$a = \frac{\alpha}{1 + \alpha} \tag{3.8}$$

This expression is motivated by the assumption of independence between *B*-columns and outlet columns.

The time congestion is calculated from Jacobaeus' (1950) ratio formula

$$E = \frac{E_m(A)}{E_n\left(\dfrac{A}{a}\right)} \tag{3.9}$$

where A is taken from (3.7) and a from (3.8).

The call congestion is for HI calculated from

$$B = \frac{E_m(A)}{E_{n-1}\left(\frac{A}{a}\right)} \tag{3.10}$$

(JACOBAEUS 1950).

Since both types of measurements are made for link systems with $n = m$, m can be inserted instead of n in (3.9) and (3.10).

The distribution $[q]$ is calculated from

$$[q] = \binom{m}{q} \cdot a^q \cdot (1-a)^{m-q} \tag{3.11}$$

a being defined in (3.8).

For the routes, the distributions $[r]$ are calculated as Erlang distributions, viz.

$$[r] = \frac{\frac{A^r}{r!}}{\sum_{\varrho=0}^{m} \frac{A^\varrho}{\varrho!}} \tag{3.12}$$

where A is defined in (3.7).

The distribution for number of busy pairs, $[p]$, is calculated from formula (2.2.32c), which for $s = 0$ is simplified to

$$[p] = \sum_{r=0}^{p} [r] \cdot \binom{m-r}{p-r} \cdot a^{p-r} \cdot (1-a)^{m-p} \tag{3.13}$$

3.1 Comparison No. 1

The first comparison is intended to give an idea of the precision of the new method, (HS I), compared with the equations of state method (ELLDIN 1956 and 1957). For this purpose a relatively small link system with $m = 4$ was used so as to permit evaluation by the latter method. For the comparison, Trial No. 22 in Wallström's paper of 1958 is used. Here $m = n = 4$, $k = v = 10$, $\alpha = 1$ and $\eta = 0.1$. The results of this measurement are given in Tables 4.11, 4.44, 4.45 and 4.51 of his paper.

For the equations of state method, (HE), the calculated values are given in the last three of the abovementioned tables. The values marked Elldin II are used. This hypothesis obviously gives the best estimate of the traffic offered to the route.

The hypotheses HM and HI have also been calculated. The results as regards the distributions are given in *Tables 3.1.1—3.1.3* and as regards the congestion values in Tables *3.1.4* and *3.1.5*.

The following traffic values have been used for hypotheses HS I, HM and HI.

	A	α
HS I:	2.373	1.0
HM:	2.353	1.0
HI:	2.297	1.0

For HS I is given the final value in the iteration for the traffic offered to the route, as described in chapter 2.

Table 3.1.1. Measured and Calculated Values for the Distributions in the B-columns

$m=4 \quad n=4 \quad k=v=10 \quad \alpha=1 \quad \eta=0.1$

5 × 10,000 calls — Measurement No. 3021 (type 1)

q	$[q]_M$	$[q]_{HE}$	$[q]_{HSI}$	$[q]_{HM}$	$[q]_{HI}$
0	0.0795	0.0758	0.0794	0.0789	0.0625
1	2880	2869	2952	2953	2500
2	3923	3877	3907	3924	3750
3	2128	2198	2052	2049	2500
4	0274	0299	0295	284	0625
$\Sigma q[q]$	1.8206	1.8410	1.8102	1.8087	2.0000
$\Sigma\|\triangle\|$		0.0189	0.0186	0.0169	0.1446
$\frac{\chi^2}{N}$		0.000672	0.000614	0.000526	0.036445

Table 3.1.2. Measured and Calculated Values for the Distributions in the C-columns

$m=4 \quad n=4 \quad k=v=10 \quad \alpha=1 \quad \eta=0.1$

5 × 10,000 calls — Measurement No. 3021 (type 1)

r	$[r]_M$	$[r]_{HE}$	$[r]_{HSI}$	$[r]_{HM}$	$[r]_{HI}$
0	0.1330	0.1427	0.1245	0.1278	0.1097
1	2880	3034	2866	2922	2520
2	3124	3180	3142	3164	2894
3	1990	1807	2039	1994	2216
4	0676	0552	0709	0643	1273
$\Sigma r[r]$	1.7802	1.7023	1.8102	1.7802	2.0048
$\Sigma\|\triangle\|$		0.0614	0.0199	0.0171	0.1645
$\frac{\chi^2}{N}$		0.006179	0.000369	0.000493	0.042181

Table 3.1.3. Measured and Calculated Values for the Distributions of Busy Pairs

$m = 4 \quad n = 4 \quad k = v = 10 \quad \alpha = 1 \quad \eta = 0.1$

5 × 10,000 calls — Measurement No. 3021 (type 1)

p	$[p]_M$	$[p]_{HE}$	$[p]_{HSI}$	$[p]_{HM}$	$[p]_{HI}$
0	0.0141	0.0133	0.0099	0.0101	0.0069
1	0956	0969	0845	0863	0589
2	2839	2781	2692	2741	2080
3	3814	3900	3965	3982	3774
4	2250	2217	2399	2313	3488
$\Sigma p[p]$	2.7076	2.7099	2.7720	2.7543	3.0023
$\Sigma\lvert\Delta\rvert$	—	0.0198	0.0600	0.0462	0.2475
$\frac{\chi^2}{N}$	—	0.000425	0.005543	0.003825	0.102135

Table 3.1.4. Measured and Calculated Values for the Time Congestion

$m = 4 \quad n = 4 \quad k = v = 10 \quad \alpha = 1 \quad \eta = 0.1$

5 × 10,000 calls — Measurement No. 3021 (type 1)

i	E_i	Δ_E	δ_E	λ_E
M	0.2250			
HE	0.2217	− 0.0033	− 0.0147	− 0.34
HS I	0.2399	+ 0.0149	+ 0.0672	+ 1.52
HM	0.2313	+ 0.0016	+ 0.0069	+ 0.16
HI	0.3488	+ 0.1238	+ 0.3549	+ 12.63

$dE_M = 0.0098$

Table 3.1.5. Measured and Calculated Values for the Call Congestion

$m = 4 \quad n = 4 \quad k = v = 10 \quad \alpha = 1 \quad \eta = 0.1$

5 × 10,000 calls — Measurement No. 3021 (type 1)

i	B_i	Δ_B	δ_B	λ_B
M	0.1620			
HE	0.1509	− 0.0111	− 0.0685	− 3.83
HS I	0.1734	+ 0.0114	+ 0.0704	+ 3.95
HM	0.1656	+ 0.0036	+ 0.0217	+ 1.24
HI	0.2544	+ 0.0924	+ 0.3632	+ 57.75

$dB_M = 0.0029$

3.2 Comparison No. 2

This comparison as well is made for a small link system with $m = 4$. The link system is unsymmetrically loaded and has two types of outlet columns. The hypotheses HS II and HI are compared with a measurement of type 2. The distributions are given in *Tables 3.2.1—3.2.3* and the congestion values in *Tables 3.2.4* and *3.2.5*.

The following traffic values have been used for hypotheses HS II and HI.

	HS II	HI
α	1.0	1.0
A_{I}	1.673	1.729
A_{II}	2.868	2.898

The values A_{I} and A_{II} for HS II are the final values in the iteration.

Table 3.2.1. Measured and Calculated Values for the Distributions in the B-columns

$m = 4 \quad n = 4 \quad k = v = 10 \quad \alpha = 1$ $\qquad v_{\mathrm{I}} = 4 \quad v_{\mathrm{II}} = 6 \quad \eta_{\mathrm{I}} = 0.07 \quad \eta_{\mathrm{II}} = 0.12$

2 × 5 × 10,000 calls $\qquad$ Measurements Nos. 3034 and 3035 (type 2)

q	$[q]_M$	$[q]_{HSII}$	$[q]_{HI}$
0	0.0775	0.0820	0.0625
1	2993	3000	2500
2	3972	3897	3750
3	2008	2002	2500
4	0251	0281	0625
$\Sigma q[q]$	1.7965	1.7924	2.0000
$\Sigma\|\triangle\|$		0.0163	0.1731
$\frac{\chi^2}{N}$		0.000715	0.046659

Table 3.2.2. Measured and Calculated Values for the Distributions in the C-columns

$m = 4 \quad n = 4 \quad k = v = 10 \quad \alpha = 1$ $\qquad v_{\mathrm{I}} = 4 \quad v_{\mathrm{II}} = 6 \quad \eta_{\mathrm{I}} = 0.07 \quad \eta_{\mathrm{II}} = 0.12$

5 × 10,000 calls per measurement $\qquad$ Measurements Nos. 3034 and 3035 (type 2)

Route type	r	$[r]_M$	$[r]_{HSII}$	$[r]_{HI}$
I	0	0.2133	0.2170	0.1832
	1	3429	3528	3168
	2	2715	2734	2739
	3	1363	1257	1579
	4	0360	0310	0682
	$\Sigma r[r]$	1.4388	1.4007	1.6111
	$\Sigma\|\triangle\|$		0.0311	0.1125
	$\frac{\chi^2}{N}$		0.002054	0.025290

Table 3.2.2. (cont.)

Route type	r	$[r]_M$	$[r]_{HSII}$	$[r]_{HI}$
II	0	0.0764	0.0859	0.0663
	1	2341	2393	1920
	2	3263	3180	2782
	3	2527	2507	2688
	4	1105	1061	1947
	$\Sigma r[r]$	2.0868	2.0518	2.3338
	$\Sigma\|\triangle\|$		0.0294	0.2006
	$\frac{\chi^2}{N}$		0.001594	0.056453

Table 3.2.3. Measured and Calculated Values for the Distributions of Busy Pairs

$m=4$ $n=4$ $k=v=10$ $\alpha=1$ $v_I=4$ $v_{II}=6$ $\eta_I=0.07$ $\eta_{II}=0.12$

5 × 10,000 calls per measurement. Measurements Nos. 3034 and 3035 (type 2)

Route type	p	$[p]_M$	$[p]_{HSII}$	$[p]_{HI}$
I	0	0.0188	0.0178	0.0115
	1	1251	1238	0854
	2	3148	3202	2560
	3	3733	3714	3805
	4	1679	1668	2667
	$\Sigma p\,[p]$	2.5464	2.5455	2.8055
	$\Sigma\|\triangle\|$		0.0108	0.2119
	$\frac{\chi^2}{N}$		0.000178	0.073449
II	0	0.0070	0.0070	0.0041
	1	0750	0674	0406
	2	2437	2393	1664
	3	3945	4005	3621
	4	2799	2858	4268
	$\Sigma p\,[p]$	2.8655	2.8907	3.1669
	$\Sigma\|\triangle\|$		0.0241	0.2938
	$\frac{\chi^2}{N}$		0.001165	0.120437

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Table 3.2.4. Measured and Calculated Values for the Time Congestion

$m = 4 \quad n = 4 \quad k = \check{v} = 10 \quad \alpha = 1 \qquad v_I = 4 \quad v_{II} = 5 \quad \eta_I = 0.07 \quad \eta_{II} = 0.12$

5 × 10,000 calls per measurement — Measurements Nos. 3034 and 3035 (type 2)

Route type	i	E_i	Δ_E	δ_E	λ_E
I	M	0.1679			
	HS II HI	0.1668 0.2667	− 0.0011 + 0.0988	− 0.0065 + 0.5883	− 0.13 + 11.49
II	M	0.2799			
	HS II HI	0.2858 0.4268	+ 0.0059 + 0.1469	+ 0.0211 + 0.5248	+ 0.59 + 14.75

$dE_{MI} = 0.0086 \qquad dE_{MII} = 0.0100$

Table 3.2.5. Measured and Calculated Values for the Call Congestion

$m = 4 \quad n = 4 \quad k = v = 10 \quad \alpha = 1 \qquad v_I = 4 \quad v_{II} = 6 \quad \eta_I = 0.07 \quad \eta_{II} = 0.12$

5 × 10,000 calls per measurement — Measurements Nos. 3034 and 3035 (type 2)

Route type	i	B_i	Δ_B	δ_B	λ_B
I	M	0.1017			
	HS II HI	0.1061 0.1717	+ 0.0044 + 0.0700	+ 0.0433 + 0.0688	+ 0.65 + 10.27
II	M	0.2021			
	HS II HI	0.2199 0.3363	+ 0.0178 + 0.1342	+ 0.0881 + 0.6643	+ 3.96 + 29.59
I + II (average for the whole link system)	M	0.1817			
	HS II HI	0.1880 0.2902	+ 0.0063 + 0.1085	+ 0.0347 + 0.5974	+ 4.50 + 51.44

$dB_{MI} = 0.0068 \qquad dB_{MII} = 0.0045 \qquad d\bar{B}_M = 0.0021$

3.3 Comparison No. 3

This comparison is for an ordinary link system with $m = 10$. The system is symmetrically loaded and the hypotheses HS I, HM and HI are compared with measurements of type 1. There are measurements for five different values of α. The results are given in *Tables 3.3.1—3.3.3* as regards the distributions and in *Tables 3.3.4—3.3.5* for the congestion values.

For HS I, HM and HI, the following values for the traffic offered to an outlet column have been used:

α	A_{HSI} erl.	A_{HM} erl.	A_{HI} erl.
0.7	4.163	4.171	4.288
1.0	5.185	5.199	5.253
1.5	6.631	6.643	6.556
2.5	9.138	9.204	9.152
5.0	14.534	13.788	13.523

The values A_{HSI} are the final ones in the iteration.

Table 3.3.1. Measured and Calculated Values for the Distributions in the B-columns

$m = 10 \quad n = 10 \quad k = 10 \quad v = 10 \quad \eta = 0.1$

5 × 10,000 calls per measurement. Measurements Nos. 3025—3029 (type 1)

α	q	$[q]_M$	$[q]_{HSI}$	$[q]_{HM}$	$[q]_{HI}$
0.7	0	0.0035	0.0051	0.0051	0.0050
	1	0318	0358	0358	0347
	2	1037	1121	1123	1094
	3	2124	2080	2083	2042
	4	2482	2524	2527	2501
	5	2183	2090	2091	2101
	6	1194	1191	1189	1226
	7	0499	0458	0454	0490
	8	0119	0111	0109	0129
	9	0010	0015	0014	0020
	10	0000	0001	0001	0001
	$\Sigma q[q]$	4.1298	4.0768	4.0732	4.1177
	$\Sigma\|\Delta\|$		0.0397	0.0385	0.0346
	$\frac{\chi^2}{N}$		0.002848	0.002928	0.002431
1.0	0	0.0002	0.0011	0.0011	0.0010
	1	0084	0108	0108	0098
	2	0402	0481	0482	0439
	3	1098	1262	1267	1172
	4	2166	2161	2170	2051
	5	2647	2513	2521	2461
	6	2163	1997	1996	2051
	7	1053	1057	1048	1172
	8	0325	0348	0338	0439
	9	0056	0060	0056	0098
	10	0004	0003	0003	0010
	$\Sigma q[q]$	4.9493	4.8798	4.8709	5.0000
	$\Sigma\|\Delta\|$		0.0613	0.0598	0.0827
	$\frac{\chi^2}{N}$		0.007007	0.006964	0.010609

Table 3.3.1. (cont.)

α	q	$[q]_M$	$[q]_{HSI}$	$[q]_{HM}$	$[q]_{HI}$
1.5	0	0.0000	0.0001	0.0001	0.0001
	1	0016	0021	0021	0016
	2	0086	0138	0139	0106
	3	0487	0536	0539	0425
	4	1208	1346	1356	1115
	5	2079	2278	2297	2007
	6	2880	2608	2621	2508
	7	2082	1958	1949	2150
	8	0943	0892	0870	1209
	9	0217	0206	0193	0403
	10	0002	0015	0013	0068
	$\Sigma q[q]$	5.8248	5.7177	5.7019	6.0000
	$\Sigma\|\triangle\|$		0.0915	0.0977	0.1198
	$\frac{\chi^2}{N}$		0.010879	0.011735	0.028192
2.5	0	0.0000	0.0000	0.0000	0.0000
	1	0000	0002	0002	0001
	2	0024	0021	0021	0010
	3	0152	0129	0131	0068
	4	0472	0520	0530	0297
	5	1218	1392	1421	0892
	6	2593	2483	2523	1859
	7	2875	2845	2855	2655
	8	1956	1919	1878	2489
	9	0635	0630	0588	1383
	10	0074	0059	0051	0346
	$\Sigma q[q]$	6.6274	6.5898	6.5612	7.1429
	$\Sigma\|\triangle\|$		0.0447	0.0525	0.3107
	$\frac{\chi^2}{N}$		0.004246	0.006060	0.138716
5	0	0.0000	0.0000	0.0000	0.0000
	1	0000	0000	0000	0000
	2	0001	0001	0001	0000
	3	0015	0013	0011	0002
	4	0112	0093	0084	0022
	5	0420	0454	0422	0130
	6	1352	1441	1384	0543
	7	2824	2858	2833	1550
	8	3193	3221	3278	2907
	9	1780	1683	1742	3230
	10	0303	0237	0245	1615
	$\Sigma q[q]$	7.5070	7.4613	7.4971	8.3333
	$\Sigma\|\triangle\|$		0.0369	0.0256	0.5525
	$\frac{\chi^2}{N}$		0.003686	0.002832	0.445752

Table 3.3.2. Measured and Calculated Values for the Distributions in the C-columns

$m = 10 \quad n = 10 \quad k = 10 \quad v = 10 \quad \eta = 0.1$

5 × 10,000 calls per measurement — Measurements Nos. 3025—3029 (type 1)

α	r	$[r]_M$	$[r]_{HSI}$	$[r]_{HM}$	$[r]_{HI}$
0.7	0	0.0105	0.0159	0.0158	0.0138
	1	0613	0660	0658	0592
	2	1319	1374	1373	1268
	3	1870	1906	1908	1813
	4	1889	1980	1986	1944
	5	1762	1643	1650	1670
	6	1185	1130	1135	1191
	7	0718	0658	0658	0730
	8	0329	0325	0320	0391
	9	0168	0130	0124	0187
	10	0043	0035	0031	0080
	$\Sigma r[r]$	4.1930	4.0768	4.0717	4.2540
	$\Sigma\lvert\triangle\rvert$		0.0567	0.0574	0.0445
	$\frac{\chi^2}{N}$		0.005851	0.006426	0.004820
1.0	0	0.0046	0.0061	0.0060	0.0053
	1	0275	0314	0314	0280
	2	0714	0814	0816	0736
	3	1486	1404	1411	1288
	4	1851	1812	1825	1692
	5	1828	1860	1875	1777
	6	1533	1574	1584	1556
	7	1121	1114	1113	1168
	8	0725	0655	0642	0767
	9	0321	0303	0283	0447
	10	0100	0089	0076	0235
	$\Sigma r[r]$	4.9439	4.8798	4.8594	5.1293
	$\Sigma\lvert\triangle\rvert$		0.0454	0.0507	0.0815
	$\frac{\chi^2}{N}$		0.003801	0.005151	0.016611
1.5	0	0.0010	0.0018	0.0018	0.0015
	1	0093	0117	0119	0100
	2	0413	0385	0393	0328
	3	0899	0846	0863	0718
	4	1446	1384	1413	1177
	5	1801	1789	1824	1543
	6	1910	1886	1911	1686
	7	1607	1635	1634	1579
	8	1042	1145	1112	1294
	9	0567	0605	0557	0943
	10	0211	0190	0159	0618
	$\Sigma r[r]$	5.6665	5.7177	5.6656	6.1511
	$\Sigma\lvert\triangle\rvert$		0.0401	0.0306	0.2092
	$\frac{\chi^2}{N}$		0.003147	0.003487	0.067167

Table 3.3.2. (cont.)

α	r	$[r]_M$	$[r]_{HSI}$	$[r]_{HM}$	$[r]_{HI}$
2.5	0	0.0009	0.0003	0.0003	0.0002
	1	0026	0027	0027	0014
	2	0206	0122	0124	0065
	3	0456	0364	0373	0197
	4	0835	0802	0826	0451
	5	1247	1378	1423	0825
	6	1756	1897	1951	1258
	7	1937	2095	2123	1645
	8	1786	1806	1776	1881
	9	1279	1119	1043	1913
	10	0464	0387	0330	1751
	$\Sigma r[r]$	6.5910	6.5898	6.5612	7.5494
	$\Sigma\|\triangle\|$		0.0903	0.1118	0.4031
	$\frac{\chi^2}{N}$		0.016774	0.025027	0.263379
5.0	0	0.0000	0.0000	0.0000	0.0000
	1	0000	0003	0004	0001
	2	0011	0019	0030	0006
	3	0126	0087	0127	0026
	4	0366	0287	0393	0089
	5	0830	0726	0924	0241
	6	1413	1431	1675	0543
	7	2134	2186	2321	1049
	8	2453	2502	2365	1774
	9	1938	1956	1606	2665
	10	0728	0802	0554	3604
	$\Sigma r[r]$	7.3780	7.4613	7.1849	8.6485
	$\Sigma\|\triangle\|$		0.0444	0.1188	0.7209
	$\frac{\chi^2}{N}$		0.006994	0.020643	0.796079

Table 3.3.3. Measured and Calculated Values for the Distributions of Busy Pairs

$m = 10 \quad n = 10 \quad k = 10 \quad v = 10 \quad \eta = 0.1$

5 × 10,000 calls per measurement — Measurements Nos. 3025—3029 (type 1)

α	p	$[p]_M$	$[p]_{HSI}$	$[p]_{HM}$	$[p]_{HI}$
0.7	0	0.0000	0.0001	0.0001	0.0001
	1	0023	0013	0013	0010
	2	0047	0088	0087	0065
	3	0291	0344	0343	0262
	4	0884	0896	0896	0725
	5	1631	1653	1657	1439
	6	2271	2227	2233	2104
	7	2248	2206	2212	2274
	8	1635	1581	1581	1792
	9	0747	0776	0769	0987
	10	0222	0215	0207	0343
	$\Sigma p[p]$	6.4070	6.3663	6.3609	6.6201
	$\Sigma\vert\Delta\vert$		0.0315	0.0305	0.1123
	$\frac{\chi^2}{N}$		0.004122	0.004061	0.021488
1.0	0	0.0000	0.0000	0.0000	0.0000
	1	0001	0002	0002	0001
	2	0015	0016	0016	0010
	3	0043	0088	0089	0059
	4	0285	0332	0334	0234
	5	0804	0878	0886	0669
	6	1791	1689	1703	1405
	7	2373	2373	2391	2185
	8	2412	2394	2399	2479
	9	1688	1626	1606	1969
	10	0588	0603	0574	0989
	$\Sigma p[p]$	7.3044	7.2571	7.2413	7.5647
	$\Sigma\vert\Delta\vert$		0.0365	0.0392	0.1530
	$\frac{\chi^2}{N}$		0.004337	0.004792	0.037188
1.5	0	0.0000	0.0000	0.0000	0.0000
	1	0000	0000	0000	0000
	2	0000	0001	0002	0001
	3	0017	0013	0013	0007
	4	0080	0075	0077	0040
	5	0314	0303	0311	0175
	6	0944	0880	0904	0568
	7	1938	1850	1896	1369
	8	2802	2766	2798	2418
	9	2548	2714	2685	3012
	10	1357	1392	1315	2410
	$\Sigma p[p]$	8.0090	8.0600	8.0293	8.4604
	$\Sigma\vert\Delta\vert$		0.0405	0.0276	0.2626
	$\frac{\chi^2}{N}$		0.002284	0.001362	0.097934

Table 3.3.3. (cont.)

α	p	$[p]_M$	$[p]_{HSI}$	$[p]_{HM}$	$[p]_{HI}$
2.5	0	0.0000	0.0000	0.0000	0.0000
	1	0000	0000	0000	0000
	2	0000	0000	0000	0000
	3	0000	0001	0001	0000
	4	0003	0008	0008	0002
	5	0072	0053	0055	0013
	6	0346	0264	0276	0078
	7	1033	0940	0981	0359
	8	2364	2317	2386	1231
	9	3384	3624	3631	3077
	10	2798	2794	2662	5241
	$\Sigma p[p]$	8.7029	8.7553	8.7221	9.2999
	$\Sigma\|\triangle\|$		0.0491	0.0550	0.4885
	$\frac{\chi^2}{N}$		0.006246	0.005406	0.466667
5.0	0	0.0000	0.0000	0.0000	0.0000
	1	0000	0000	0000	0000
	2	0000	0000	0000	0000
	3	0000	0000	0000	0000
	4	0000	0000	0000	0000
	5	0006	0003	0004	0000
	6	0061	0033	0042	0003
	7	0309	0248	0298	0029
	8	1302	1227	1388	0250
	9	3778	3627	3773	1657
	10	4544	4860	4496	8063
	$\Sigma p[p]$	9.2417	9.3023	9.2370	9.7747
	$\Sigma\|\triangle\|$		0.0634	0.0171	0.7036
	$\frac{\chi^2}{N}$		0.007318	0.001632	1.250236

Table 3.3.4. Measured and Calculated Values for the Time Congestion

$m = 10 \quad n = 10 \quad k = 10 \quad v = 10 \quad \eta = 0.1$

5 × 10,000 calls per measurement — Measurements Nos. 3025—3029 (type 1)

α	i	E_i	dE_M	$\triangle_E$	δ_E	λ_E
0.7	M	0.0222	0.0024			
	HS I	0.0215		− 0.0007	− 0.0315	− 0.29
	HM	0.0207		− 0.0015	− 0.0725	− 0.62
	HI	0.0343		+ 0.0121	+ 0.5450	+ 5.04
1.0	M	0.0588	0.0027			
	HS I	0.0603		+ 0.0015	+ 0.0255	+ 0.55
	HM	0.0574		− 0.0014	− 0.0238	− 0.51
	HI	0.0990		+ 0.0402	+ 0.6837	+ 14.9

Table 3.3.4. (cont.)

α	i	E_i	dE_M	Δ_E	δ_E	λ_E
1.5	M	0.1357	0.0062			
	HS I	0.1392		+ 0.0035	+ 0.0258	+ 0.56
	HM	0.1315		− 0.0042	− 0.0310	− 0.68
	HI	0.2411		+ 0.1054	+ 0.7767	+ 17.0
2.5	M	0.2798	0.0143			
	HS I	0.2794		− 0.0004	− 0.0014	− 0.03
	HM	0.2662		− 0.0136	− 0.0486	− 0.95
	HI	0.5241		+ 0.2443	+ 0.8731	+ 17.1
5.0	M	0.4544	0.0113			
	HS I	0.4860		+ 0.0316	+ 0.0695	+ 2.78
	HM	0.4496		− 0.0048	− 0.0106	− 0.42
	HI	0.8063		+ 0.3519	+ 0.7744	+ 31.1

Table 3.3.5. Measured and Calculated Values for the Call Congestion

$m = 10 \quad n = 10 \quad k = 10 \quad v = 10 \quad \eta = 0.1$

5 × 10,000 calls per measurement — Measurements Nos. 3025—3029 (type 1)

α	i	B_i	dB_M	Δ_B	δ_B	λ_B
0.7	M	0.0164	0.0011			
	HS I	0.0167		+ 0.0003	+ 0.0182	+ 0.3
	HM	0.0160		− 0.0004	− 0.0244	− 0.4
	HI	0.0273		+ 0.0109	+ 0.6646	+ 9.91
1.0	M	0.0419	0.0013			
	HS I	0.0469		+ 0.0050	+ 0.1193	+ 3.8
	HM	0.0445		+ 0.0026	+ 0.0621	+ 2.0
	HI	0.0793		+ 0.0374	+ 0.8926	+ 28.8
1.5	M	0.1012	0.0013			
	HS I	0.1099		+ 0.0087	+ 0.0860	+ 6.5
	HM	0.1032		+ 0.0020	+ 0.0198	+ 1.5
	HI	0.1959		+ 0.0947	+ 0.9358	+ 72.8
2.5	M	0.2134	0.0013			
	HS I	0.2270		+ 0.0136	+ 0.0637	+ 10.4
	HM	0.2156		+ 0.0022	+ 0.0103	+ 1.7
	HI	0.4472		+ 0.2338	+ 1.0956	+ 179.8
5.0	M	0.3926	0.0014			
	HS I	0.4122		+ 0.0196	+ 0.0499	+ 14.0
	HM	0.3727		− 0.0199	− 0.0507	− 14.2
	HI	0.7234		+ 0.3308	+ 0.8426	+ 236.3

3.4 Comparison No. 4

The same link system, with $m = 10$, as in comparison No. 3 is here asymmetrically loaded and the hypotheses HS II and HI are compared with five measurements of type 2. One of the $v = 10$ outlet columns is offered a varied part of the total traffic:

$$\eta_{\mathrm{I}} = 0.10,\ 0.12,\ 0.15 \text{ and } 0.25,$$

while α remains constant so that the effect of asymmetrical loading can be studied. The result of the comparison is given in *Tables 3.4.1—3.4.3* for the distributions and in *Tables 3.4.4* and *3.4.5* for the congestion values. Unfortunately the measurements do not give all information about the outlet columns of the second type. The comparisons for the distributions in the outlet columns $[r]$, for the busy pairs $[p]$ and for the time congestion E are consequently limited to the first type of outlet column. For the same reason, it has not been possible to calculate the traffic offered to the second type of outlet column, A_{II}, from the measurement values, so that the call congestion for hypothesis HI could not be calculated. However, the comparisons made for the outlet column of type I give enough information to form an opinion on the precision of the hypotheses.

The following values for the traffic offered to the outlet column of type I have been used:

η_I	$A_{I,\,HSII}$	$A_{I,\,HI}$
0.10	5.185	5.253
0.12	6.226	6.477
0.15	7.813	8.125
0.20	10.548	10.821
0.25	13.421	13.810

The values for HS II are the final ones for the iteration.

Table 3.4.1. *Measured and Calculated Values for the Distributions in the B-columns*

$m = 10 \quad n = 10 \quad k = 10 \quad v = 10 \quad v_I = 1 \quad v_{II} = 9 \quad \alpha = 1$

5 × 10,000 calls per measurement — Measurements Nos. 3026 (type 1), 3030—3033 (type 2)

η_I	η_{II}	q	$[q]_M$	$[q]_{HSII}$	$[q]_{HI}$
0.1	0.1	0	0.0002	0.0011	0.0010
		1	0084	0108	0098
		2	0402	0481	0439
		3	1098	1262	1172
		4	2166	2161	2051
		5	2647	2513	2461
		6	2163	1997	2051
		7	1053	1057	1172
		8	0325	0348	0439
		9	0056	0060	0098
		10	0004	0003	0010
		$\Sigma q[q]$	4.9493	4.8798	5.0000
		$\Sigma\lvert\Delta\rvert$		0.0613	0.0827
		$\frac{\chi^2}{N}$		0.007007	0.010606
0.12	0.097778	0	0.0012	0.0011	0.0010
		1	0129	0109	0098
		2	0496	0482	0439
		3	1176	1265	1172
		4	2036	2164	2051
		5	2590	2513	2461
		6	2030	1994	2051
		7	1105	1054	1172
		8	0352	0346	0439
		9	0070	0060	0098
		10	0004	0003	0010
		$\Sigma q[q]$	4.9148	4.8764	5.0000
		$\Sigma\lvert\Delta\rvert$		0.0433	0.0626
		$\frac{\chi^2}{N}$		0.002557	0.007304
0.15	0.094444	0	0.0012	0.0011	0.0010
		1	0102	0111	0098
		2	0435	0492	0439
		3	1196	1282	1172
		4	2157	2180	2051
		5	2473	2513	2461
		6	2143	1978	2051
		7	1074	1036	1172
		8	0346	0336	0439
		9	0062	0057	0098
		10	0001	0003	0010
		$\Sigma q[q]$	4.9258	4.8584	5.0000
		$\Sigma\lvert\Delta\rvert$		0.0436	0.0480
		$\frac{\chi^2}{N}$		0.003128	0.005996

Table 3.4.1. (cont.)

η_I	η_{II}	q	$[q]_M$	$[q]_{HSII}$	$[q]_{HI}$
0.20	0.088889	0	0.0010	0.0012	0.0010
		1	0144	0121	0098
		2	0579	0525	0439
		3	1459	1342	1172
		4	2250	2231	2051
		5	2450	2510	2461
		6	1776	1921	2051
		7	0975	0976	1172
		8	0301	0307	0439
		9	0056	0051	0098
		10	0001	0003	0010
		$\Sigma q[q]$	4.7331	4.7975	5.0000
		$\Sigma\|\triangle\|$		0.0434	0.1344
		$\frac{\chi^2}{N}$		0.003493	0.029532
0.25	0.083333	0	0.0010	0.0015	0.0010
		1	0140	0138	0098
		2	0537	0580	0439
		3	1472	1435	1172
		4	2325	2302	2051
		5	2494	2493	2461
		6	1918	1832	2051
		7	0819	0892	1172
		8	0238	0268	0439
		9	0047	0042	0098
		10	0002	0002	0010
		$\Sigma q[q]$	4.6975	4.706$\underline{5}$	5.0000
		$\Sigma\|\triangle\|$		0.0305	0.1493
		$\frac{\chi^2}{N}$		0.002863	0.039361

Table 3.4.2. Measured and Calculated Values for the Distribution in the C-column, type I

$m = 10 \quad n = 10 \quad k = 10 \quad v = 10 \quad v_I = 1 \quad v_{II} = 9 \quad \alpha = 1$

5 × 10,000 calls per measurement — Measurements Nos. 3026 (type 1), 3030—3033 (type 2)

η_I	η_{II}	r_I	$[r_I]_M$	$[r_I]_{HSII}$	$[r_I]_{HI}$
0.1	0.1	0	0.0046	0.0061	0.0053
		1	0275	0314	0280
		2	0714	0814	0736
		3	1486	1404	1288
		4	1851	1812	1692
		5	1828	1860	1777
		6	1533	1574	1556
		7	1121	1114	1168
		8	0725	0655	0767
		9	0321	0303	0447
		10	0100	0089	0235
		$\Sigma r[r]$	4.9439	4.8798	5.1293
		$\Sigma\|\triangle\|$		0.0454	0.0815
		$\frac{\chi^2}{N}$		0.003801	0.016609
0.12	0.097778	0	0.0012	0.0023	0.0016
		1	0106	0144	0107
		2	0332	0449	0345
		3	0829	0930	0746
		4	1478	1441	1207
		5	1809	1776	1564
		6	1758	1805	1688
		7	1548	1535	1562
		8	1153	1084	1264
		9	0723	0601	0910
		10	0253	0212	0589
		$\Sigma r[r]$	5.7858	5.6252	6.0951
		$\Sigma\|\triangle\|$		0.0629	0.1335
		$\frac{\chi^2}{N}$		0.009670	0.035280
0.15	0.094444	0	0.0005	0.0006	0.0004
		1	0025	0045	0030
		2	0138	0177	0122
		3	0426	0461	0329
		4	0798	0896	0669
		5	1308	1387	1087
		6	1656	1770	1473
		7	1998	1890	1709
		8	1924	1677	1736
		9	1175	1170	1567
		10	0548	0519	1273
		$\Sigma r[r]$	6.6676	6.5296	7.0905
		$\Sigma\|\triangle\|$		0.0775	0.2246
		$\frac{\chi^2}{N}$		0.008706	0.071737

Table 3.4.2. (cont.)

η_{I}	η_{II}	r_{I}	$[r_{\mathrm{I}}]_M$	$[r_{\mathrm{I}}]_{HSII}$	$[r_{\mathrm{I}}]_{HI}$
0.2	0.088889	0	0.0001	0.0001	0.0000
		1	0006	0007	0004
		2	0027	0038	0024
		3	0140	0135	0088
		4	0335	0354	0237
		5	0680	0740	0513
		6	1174	1277	0925
		7	1785	1846	1430
		8	2244	2220	1935
		9	2245	2106	2326
		10	1363	1276	2517
		$\Sigma r[r]$	7.6548	7.5661	8.0974
		$\Sigma\lvert\Delta\rvert$		0.0510	0.2471
		$\frac{\chi^2}{N}$		0.003509	0.086438
0.25	0.083333	0	0.0000	0.0000	0.0000
		1	0000	0001	0001
		2	0003	0009	0005
		3	0027	0041	0023
		4	0111	0136	0081
		5	0307	0362	0223
		6	0755	0796	0514
		7	1431	1469	1013
		8	2255	2263	1749
		9	2785	2759	2684
		10	2325	2163	3707
		$\Sigma r[r]$	8.2973	8.2126	8.6912
		$\Sigma\lvert\Delta\rvert$		0.0376	0.2769
		$\frac{\chi^2}{N}$		0.003823	0.099612

Table 3.4.3. *Measured and Calculated Values for the Distribution of Busy Pairs, C-column of type I*

$m = 10 \quad n = 10 \quad k = 10 \quad v = 10 \quad v_I = 1 \quad v_{II} = 9 \quad \alpha = 1$

5 × 10,000 calls per measurement — Measurements Nos. 3026 (type 1), 3030—3033 (type 2)

η_I	η_{II}	p_I	$[p_I]_M$	$[p_I]_{HSII}$	$[p_I]_{HI}$
0.1	0.1	0	0.0000	0.0000	0.0000
		1	0001	0002	0001
		2	0015	0016	0010
		3	0043	0088	0059
		4	0285	0332	0234
		5	0804	0878	0669
		6	1791	1689	1405
		7	2373	2373	2185
		8	2412	2394	2479
		9	1688	1626	1969
		10	0588	0603	0989
		$\Sigma p[p]$	7.3044	7.2571	7.5647
		$\Sigma\|\triangle\|$		0.0365	0.1530
		$\frac{\chi^2}{N}$		0 004337	0.037192
0.12	0.097778	0	0.0000	0.0000	0.0000
		1	0000	0001	0000
		2	0006	0008	0004
		3	0042	0049	0026
		4	0170	0204	0118
		5	0570	0607	0390
		6	1335	1327	0961
		7	2098	2154	1782
		8	2530	2562	2475
		9	2181	2108	2506
		10	1067	0980	1737
		$\Sigma p[p]$	7.6909	7.6325	8.0475
		$\Sigma\|\triangle\|$		0.0337	0.1990
		$\frac{\chi^2}{N}$		0.002259	0.062023
0.15	0.094444	0	0.0000	0.0000	0.0000
		1	0001	0000	0000
		2	0002	0003	0001
		3	0022	0020	0009
		4	0078	0099	0047
		5	0315	0343	0186
		6	0822	0888	0550
		7	1621	1741	1253
		8	2569	2564	2194
		9	2775	2692	2916
		10	1794	1650	2844
		$\Sigma p[p]$	8.1708	8.0932	8.5453
		$\Sigma\|\triangle\|$		0.0471	0.2381
		$\frac{\chi^2}{N}$		0.003563	0.081425

Table 3.4.3. (cont.)

η_{I}	η_{II}	p_{I}	$[p_{I}]_{M}$	$[p_{I}]_{HSII}$	$[p_{I}]_{HI}$
0.20	0.088889	0	0.0000	0.0000	0.0000
		1	0000	0000	0000
		2	0000	0001	0000
		3	0004	0005	0002
		4	0029	0031	0012
		5	0134	0135	0059
		6	0429	0443	0225
		7	1089	1128	0675
		8	2184	2221	1600
		9	3205	3218	3006
		10	2926	2819	4421
		$\Sigma p[p]$	8.6573	8.6284	9.0487
		$\Sigma\|\triangle\|$		0.0215	0.2990
		$\frac{\chi^2}{N}$		0.000786	0.131217
0.25	0.083333	0	0.0000	0.0000	0.0000
		1	0000	0000	0000
		2	0000	0000	0000
		3	0002	0002	0000
		4	0008	0011	0003
		5	0070	0057	0020
		6	0190	0227	0094
		7	0675	0718	0358
		8	1793	1785	1103
		9	3270	3339	2769
		10	3992	3861	5653
		$\Sigma p[p]$	8.9950	8.9664	9.3454
		$\Sigma\|\triangle\|$		0.0304	0.3322
		$\frac{\chi^2}{N}$		0.001832	0.152640

Table 3.4.4. Measured and Calculated Values for the Time Congestion, C-column of type I

$m = 10$ $n = 10$ $k = 10$ $v = 10$ $v_I = 1$ $v_{II} = 9$ $\alpha = 1$

5 × 10,000 calls per measurement   Measurements Nos. 3026 (type 1) 3030—3033 (type 2)

η_I	i	E_i	dE_M	Δ_E	δ_E	λ_E
0.10	M	0.0588	0.0027			
	HS II	0.0603		+ 0.0015	+ 0.0255	+ 0.55
	HI	0.0990		+ 0.0402	+ 0.6837	+ 14.9
0.12	M	0.1067	0.0034			
	HS II	0.0980		− 0.0087	− 0.0816	− 2.55
	HI	0.1737		+ 0.0670	+ 0.6279	+ 19.7
0.15	M	0.1794	0.0068			
	HS II	0.1650		− 0.0144	− 0.0802	− 2.13
	HI	0.2844		+ 0.1050	+ 0.5853	+ 15.4
0.20	M	0.2926	0.0109			
	HS II	0.2819		− 0.0107	− 0.0366	− 0.98
	HI	0.4421		+ 0.1495	+ 0.5109	+ 13.7
0.25	M	0.3992	0.0089			
	HS II	0.3861		− 0.0131	− 0.0328	− 1.46
	HI	0.5653		+ 0.1661	+ 0.4161	+ 18.7

Table 3.4.5. I. Measured and Calculated Values for the Call Congestion, C-column of type I

$m = 10$ $n = 10$ $k = 10$ $v = 10$ $v_I = 1$ $v_{II} = 9$ $\alpha = 1$

5 × 10,000 calls per measurement   Measurements Nos. 3026 (type 1), 3030—3033 (type 2)

η_I	i	B_i	dB_M	Δ_B	δ_B	λ_B
0.10	M	0.0419	0.0013			
	HS II	0.0469		+ 0.0050	+ 0.1193	+ 3.78
	HI	0.0793		+ 0.0374	+ 0.8926	+ 28.8
0.12	M	0.0798	0.0053			
	HS II	0.0807		+ 0.0009	+ 0.0113	+ 0.17
	HI	0.1486		+ 0.0688	+ 0.8622	+ 13.0
0.15	M	0.1529	0.0029			
	HS II	0.1432		− 0.0097	− 0.0634	− 3.38
	HI	0.2553		+ 0.1024	+ 0.6697	+ 35.3
0.20	M	0.2617	0.0068			
	HS II	0.2568		− 0.0049	− 0.0187	− 0.72
	HI	0.4120		+ 0.1503	+ 0.5743	+ 22.1
0.25	M	0.3694	0.0052			
	HS II	0.3613		− 0.0081	− 0.0219	− 1.55
	HI	0.5377		+ 0.1683	+ 0.4556	+ 32.4

Table 3.4.5. II. Call Congestion. Routes of type II

η_{I}	η_{II}	i	B_i	dB_M	$\triangle_B$	δ_B	λ_B
0.10	0.10	M	0.0419	0.0013			
		HS II	0.0469		+ 0.0050	+ 0.1193	+ 3.78
		HI	0.0793		+ 0.0374	+ 0.8926	+ 28.8
0.12	0.097778	M	0.0378	0.0012			
		HS II	0.0438		+ 0.0060	+ 0.1588	+ 4.86
		HI	—		—	—	—
0.15	0.094444	M	0.0370	0.0023			
		HS II	0.0395		+ 0.0025	+ 0.0676	+ 1.07
		HI	—		—	—	—
0.20	0.088889	M	0.0291	0.0086			
		HS II	0.0331		+ 0.0040	+ 0.1374	+ 4.63
		HI	—		—	—	—
0.25	0.083333	M	0.0214	0.0016			
		HS II	0.0274		+ 0.0060	+ 0.2806	+ 3.68
		HI	—		—	—	—

Table 3.4.5. III. Call Congestion. Average Values for All Calls

η_{I}	η_{II}	i	$\bar{B}_i$	$d\bar{B}_M$	$\triangle_{\bar{B}}$	$\delta_{\bar{B}}$	$\lambda_{\bar{B}}$
0.10	0.10	M	0.0419	0.0013			
		HS II	0.0469		+ 0.0050	+ 0.1193	+ 3.78
		HI	0.0793		+ 0.0374	+ 0.8926	+ 28.8
0.12	0.097778	M	0.0429	0.0035			
		HS II	0.0482		+ 0.0053	+ 0.1237	+ 1.51
		HI	—				
0.15	0.094444	M	0.0542	0.0019			
		HS II	0.0551		+ 0.0009	+ 0.0166	+ 0.48
		HI	—				
0.20	0.088889	M	0.0756	0.0014			
		HS II	0.0779		+ 0.0023	+ 0.0304	+ 1.66
		HI	—				
0.25	0.083333	M	0.1088	0.0017			
		HS II	0.1109		+ 0.0021	+ 0.0193	+ 1.22
		HI	—				

3.5 Result of Comparisons as regards the Calculated Distributions

For judging the precision of the calculated distributions the deviation measures $\Sigma|\triangle_\nu|$ and $\frac{\chi^2}{N}$, defined in (3.3) and (3.4), have been used.

For the symmetrically loaded link system (Comparisons Nos. 1 and 3) the values for $\Sigma|\triangle_\nu|$ are listed in *Tables 3.5.1* and *3.5.2* and the values for $\frac{\chi^2}{N}$ in *Tables 3.5.3* and *3.5.4.*

For unsymmetrically loaded link systems (Comparisons Nos. 2 and 4) the values are given in *Tables 3.5.5* and *3.5.6* for $\Sigma|\triangle|$ and in *Tables 3.5.7* and *3.5.8* for $\frac{\chi^2}{N}$.

Table 3.5.1. Values for $\Sigma|\triangle|$ for Comparison No. 1 (cf. tables 3.1.1—3.1.3)

$m=4 \quad n=4 \quad k=10 \quad \mathrm{v}=10 \quad \eta=0.1 \quad \alpha=1$

ν	$\Sigma\lvert\triangle_\nu\rvert \cdot 10^4$			
	HE	HS I	HM	HI
q	189	186	169	1446
r	614	199	171	1645
p	198	600	462	2475
Σ	1001	985	802	5566

Table 3.5.2. Values for $\Sigma|\triangle|$ for Comparison No. 3 (cf. tables 3.3.1—3.3.3)

$m=10 \quad n=10 \quad k=10 \quad \mathrm{v}=10 \quad \eta=0.1$

α	$\Sigma\lvert\triangle_q\rvert \cdot 10^4$			$\Sigma\lvert\triangle_r\rvert \cdot 10^4$			$\Sigma\lvert\triangle_p\rvert \cdot 10^4$		
	HS I	HM	HI	HS I	HM	HI	HS I	HM	HI
0.7	397	385	346	567	574	445	315	305	1123
1.0	613	598	827	454	507	815	365	392	1530
1.5	915	977	1198	401	306	2092	405	276	2626
2.5	447	525	3107	903	1118	4031	491	550	4885
5.0	369	256	5525	444	1188	7209	634	171	7036
Σ	2741	2741		2769	3693		2210	1694	

Table 3.5.3. *Values for* $\frac{\chi^2}{N}$ *for Comparison No. 1 (cf. tables 3.1.1—3.1.3)*

$m = 4 \quad n = 4 \quad k = 10 \quad v = 10 \quad \eta = 0.1 \quad \alpha = 1$

ν	$\frac{\chi^2}{N} \cdot 10^4$			
	HE	HS I	HM	HI
q	7	6	5	364
r	62	9	5	422
p	4	55	38	1021
Σ	73	70	48	1807

Table 3.5.4. *Values for* $\frac{\chi^2}{N}$ *for Comparison No. 3 (cf. tables 3.3.1—3.3.3)*

$m = 10 \quad n = 10 \quad k = 10 \quad v = 10 \quad \eta = 0.1$

α	$\left(\frac{\chi^2}{N}\right)_q \cdot 10^4$			$\left(\frac{\chi^2}{N}\right)_r \cdot 10^4$			$\left(\frac{\chi^2}{N}\right)_p \cdot 10^4$		
	HS I	HM	HI	HS I	HM	HI	HS I	HM	HI
0.7	28	29	24	59	64	48	41	41	215
1.0	70	70	106	38	52	166	43	48	372
1.5	109	117	282	31	35	672	23	14	979
2.5	42	61	1387	168	250	2634	62	54	4667
5.0	37	28	4458	70	206	7961	73	16	12502
Σ	286	305	6257	366	607	11481	242	173	18735

Table 3.5.5. *Values for* $\Sigma|\triangle|$ *for Comparison No. 2 (cf. tables 3.2.1—3.2.3)*

$m = 4 \quad n = 4 \quad k = 10 \quad v = 10 \quad \alpha = 1$

$v_{I} = 4 \quad v_{II} = 6 \quad \eta_{I} = 0.07 \quad \eta_{II} = 0.12$

ν	$\Sigma|\triangle\nu| \cdot 10^4$	
	HS II	HI
q	163	1731
r_{I}	311	1125
r_{II}	294	2006
p_{I}	108	2119
p_{II}	241	2938

Table 3.5.6. *Values for $\Sigma|\triangle|$ for Comparison No. 4 (cf. tables 3.4.1—3.4.3)*

$m = 10 \quad n = 10 \quad k = 10 \quad \nu = 10 \quad \nu_I = 1 \quad \nu_{II} = 9 \quad \alpha = 1$

η_I	$\Sigma|\triangle_q| \cdot 10^4$		$\Sigma|\triangle_{r_I}| \cdot 10^4$		$\Sigma|\triangle_{p_I}| \cdot 10^4$	
	HS II	HI	HS II	HI	HS II	HI
0.10	613	827	454	815	365	1530
0.12	433	626	629	1335	337	1990
0.15	436	480	775	2246	471	2381
0.20	434	1344	510	2471	215	2990
0.25	305	1493	376	2769	304	3322

Table 3.5.7. *Values for $\frac{\chi^2}{N}$ for Comparison No. 2 (cf. tables 3.2.1—3.2.3)*

$m = 4 \quad n = 4 \quad k = 10 \quad \nu = 10 \quad \alpha = 1$

$\nu_I = 4 \quad \nu_{II} = 6 \quad \eta_I = 0.07 \quad \eta_{II} = 0.12$

v	$\frac{\chi^2}{N} \cdot 10^4$	
	HS II	HI
q	7	467
r_I	21	253
r_{II}	16	565
p_I	2	734
p_{II}	12	1204

Table 3.5.8. *Values for $\frac{\chi^2}{N}$ for Comparison No. 4 (cf. tables 3.4.1—3.4.3)*

$m = 10 \quad n = 10 \quad k = 10 \quad \nu = 10 \quad \nu_I = 1 \quad \nu_{II} = 9 \quad \alpha = 1$

η_I	$\left(\frac{\chi^2}{N}\right)_q \cdot 10^4$		$\left(\frac{\chi^2}{N}\right)_{r_I} \cdot 10^4$		$\left(\frac{\chi^2}{N}\right)_{p_I} \cdot 10^4$	
	HS II	HI	HS II	HI	HS II	HI
0.10	71	106	38	166	43	372
0.12	26	73	97	353	23	620
0.15	31	60	87	717	36	814
0.20	35	295	35	864	8	1312
0.25	29	394	38	996	18	1526

From *Table 3.5.1* and *3.5.3* it is seen that the equations of state method, HE, and the new methods HS I and HM have about the same precision as regards the distributions [q], [r] and [p]. HE evidently gives the best estimation for the resulting distribution [p], but HS I and HM have a fully acceptable precision. It can consequently be concluded that HS I and HM are acceptable substitutions for HE, although they may have a little less precision.

From *Tables 3.5.1—3.5.4* can be seen that HM is perhaps a little more accurate than HS I. This can be explained by the fact that, in HM, the values of the traffic offered to the routes are very good, since they are measured values. Since it is not always possible to use measured values, the very good precision of HM must be considered to be incidental and dependent on the correctness of the available values of the traffic offered to the route.

From *Tables 3.5.1—3.5.4* it is seen that HI has less precision than the other hypotheses. It is also seen that the accuracy for HI decreases when the traffic increases. This is in good agreement with the observation that the dependence in a link system is specially significant for large traffic values, i.e. when the congestion is large. (See also *figs. 3.1—3.3*).

In *Tables 3.5.5—3.5.8* a comparison between HS II and HI shows that HS II gives better values than HI. The improvement is specially important for the distributions [q] and [p]. It is also observed that HS II gives about the same precision here as did HS I for the symmetrical case. Application of the method HS has consequently about the same accuracy for symmetrical and asymmetrical cases.

In good agreement with the observations for HI in the symmetrical case, it is also observed that HI gives less precision when the traffic is increased, here being represented by growing values for η_{I}.

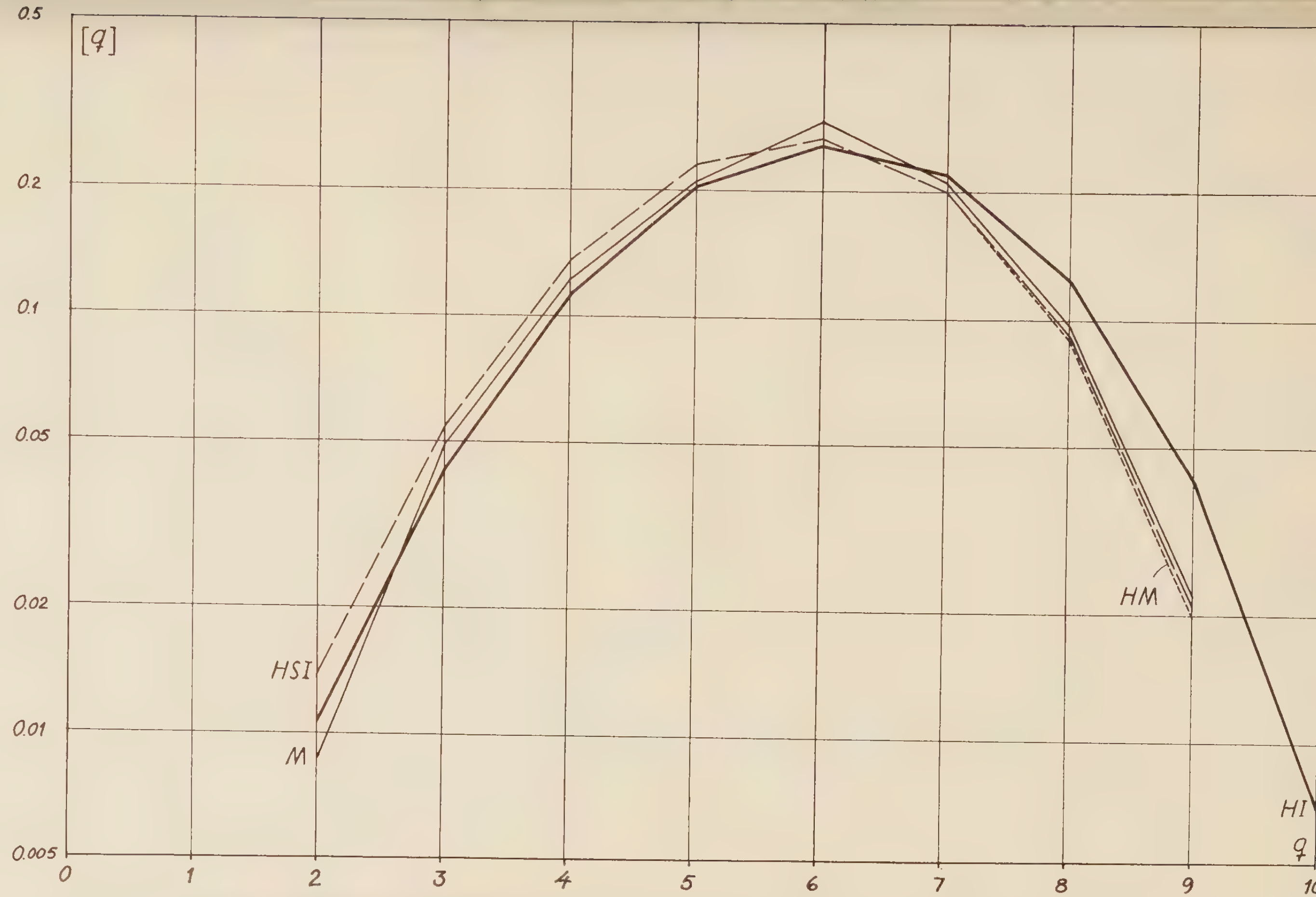

Fig. 3.1. Measured and calculated values for the distribution in the *B*-columns as given in Table 3.3.1.
($m = 10$, $n = 10$, $k = 10$, v - 10, $\eta = 0.1$, $\alpha = 1.5$).

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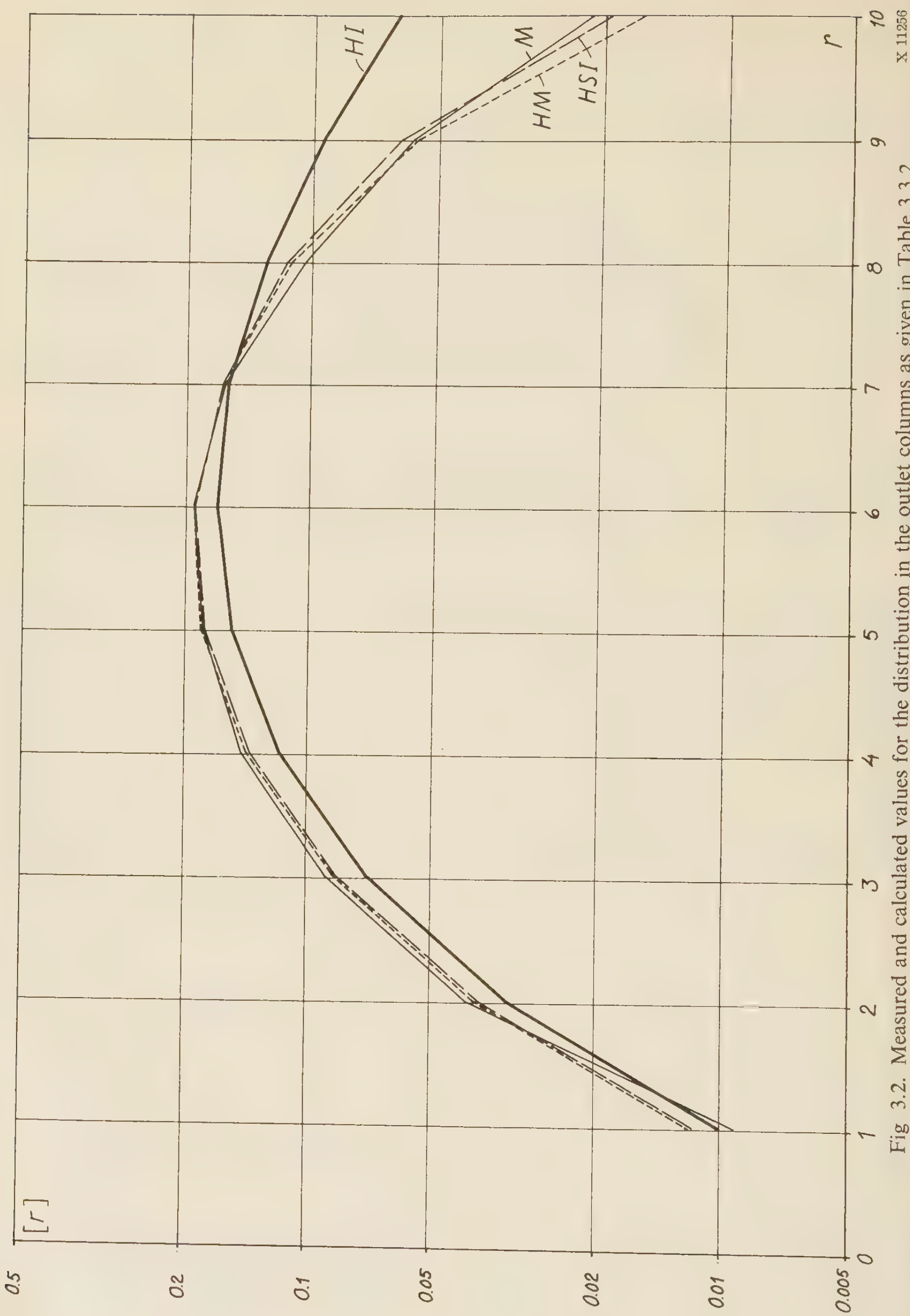

Fig 3.2. Measured and calculated values for the distribution in the outlet columns as given in Table 3.3.2. ($m = 10$, $n = 10$, $k = 10$, $v = 10$, $\eta = 0.1$, $\alpha = 1.5$).

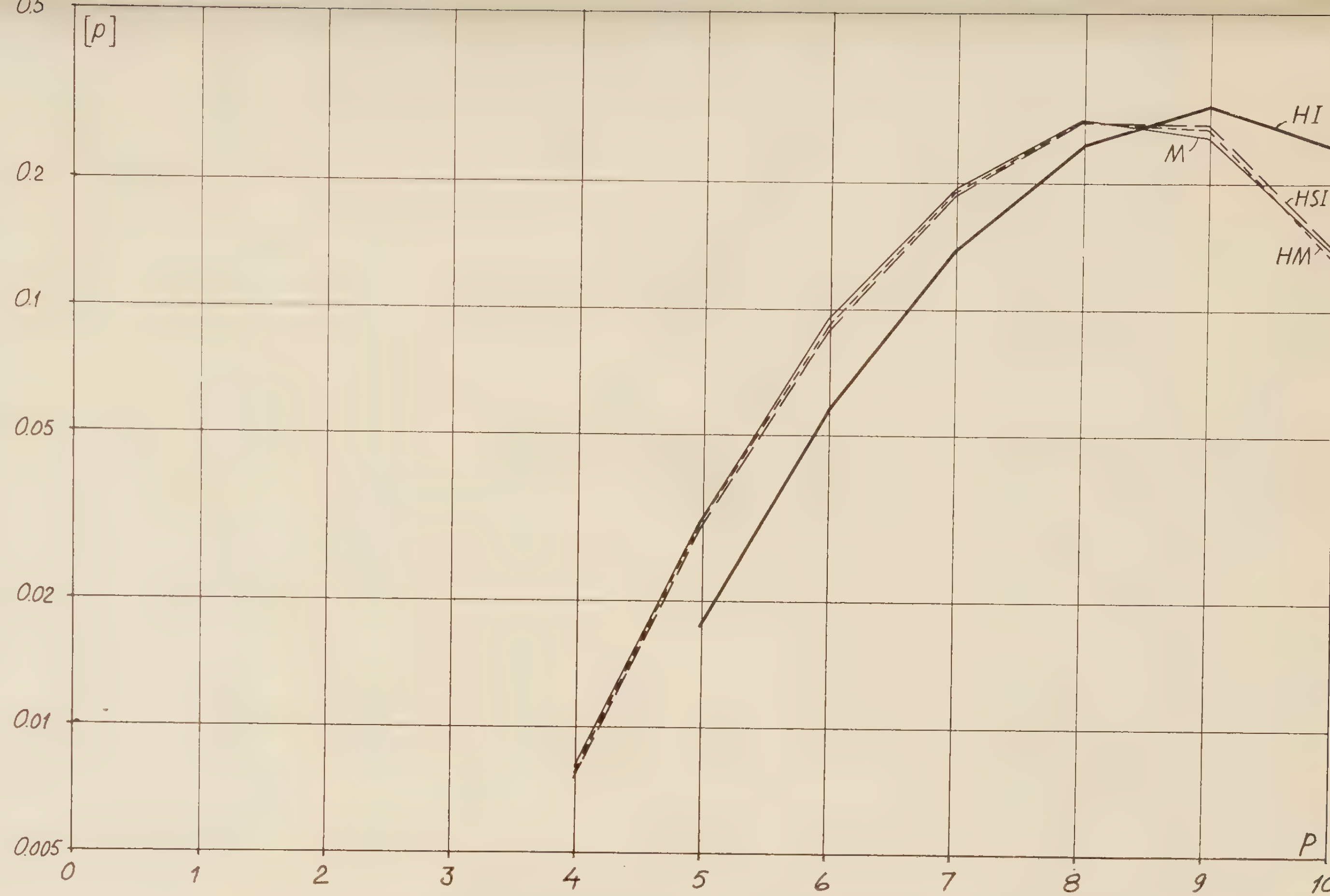

Fig. 3.3. Measured and calculated values for the distribution of busy pairs as given in Table 3.3.3.
($m = 10$, $n = 10$, $k = 10$, v = 10, $\eta = 0.1$, $\alpha = 1.5$).

3.6 Result of Comparisons as regards the Calculated Congestion Values

For describing the accuracy of the calculated congestion values, the precision measures $\triangle_E$, δ_E and λ_E, defined in (3.5)—(3.6), have been used.

The precision measures for the symmetrical case are given in *Table 3.1.4* for $m = 4$ and in *Table 3.3.4* for $m = 10$ for the time congestion and in *Tables 3.1.5* and *3.3.5*, respectively, for the call congestion.

For $m = 4$ it can be seen that the equations of state method, HE, has a little better precision than HS I and HM as regards the time congestion but about the same precision as the two

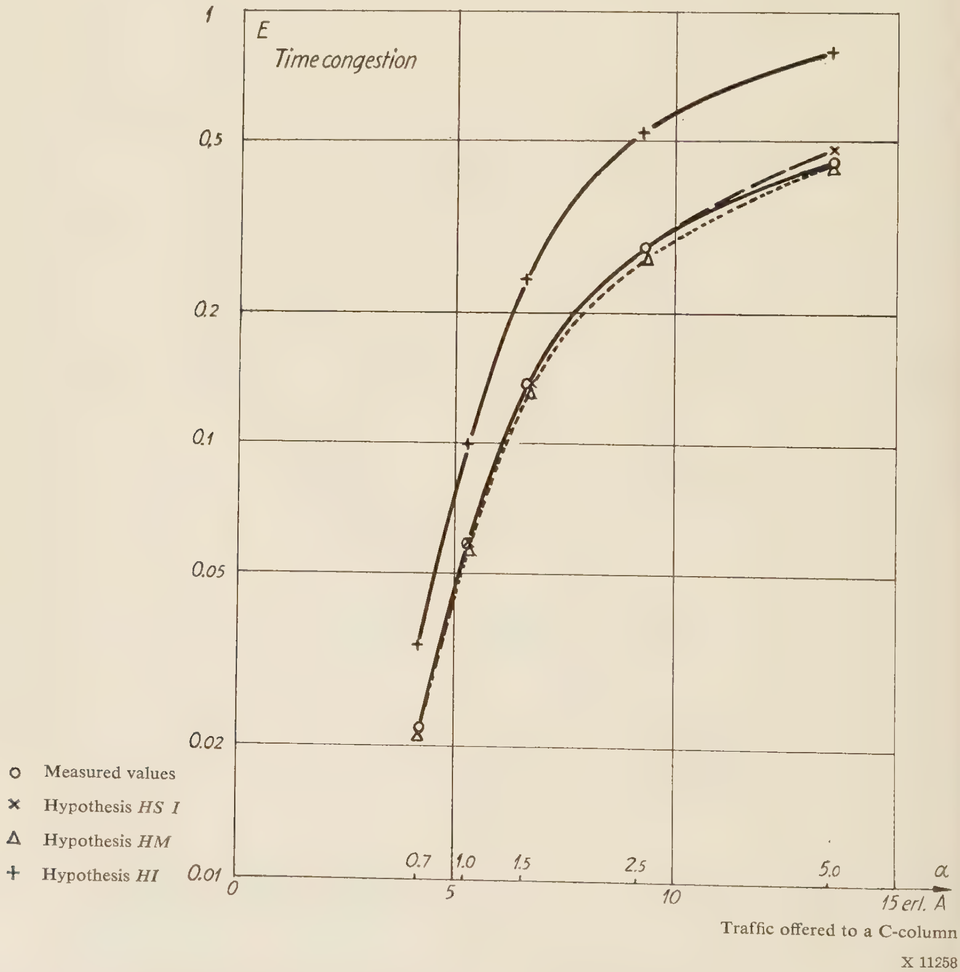

Fig. 3.4. Measured and calculated values for the time congestion as given in Table 3.3.4. ($m = 10$, $n = 10$, $k = 10$, $v = 10$, $\eta = 0.1$).

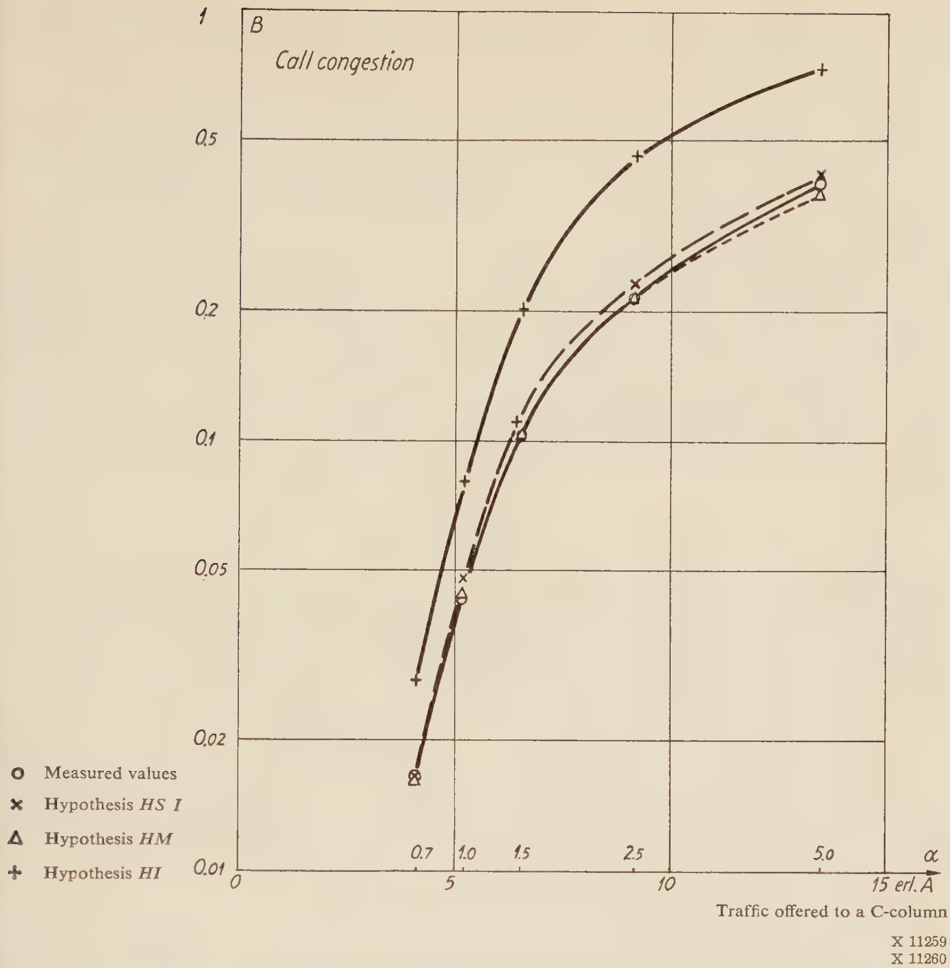

Fig. 3.5. Measured and calculated values for the call congestion as given in Table 3.3.5. ($m = 10$, $n = 10$, $k = 10$, $v = 10$, $\eta = 0.1$).

abovementioned hypotheses as regards call congestion. The precision for HE, HS I and HM is, as expected, better than that for HI.

For $m = 10$, HS I and HM are of about the same precision, HM perhaps giving a little better values for the call congestion. Both hypotheses show a clear improvement compared with hypothesis HI.

For $m = 4$, *Tables 3.2.4* and *3.2.5* show that hypothesis HS II gives about the same precision as did HS I for the symmetrical case. It is a clear improvement compared with hypothesis HI. The asymmetrical case can consequently be treated with the same precision as the symmetrical case both for congestion values and for the distributions.

For $m = 10$, *Tables 3.4.4* and *3.4.5* show also that HS II gives about the same precision as HS I for the symmetrical case. Here, again, the values of HS II show a clear improvement compared with HI.

It is seen from the abovementioned tables that HI almost always gives too large congestion values and that the deviations from measured values are more significant the larger the congestion.

From the measurements it can be seen that HI has a very clear tendency to give too large congestion values. For HS I, HS II and HM it is more difficult to define a clear tendency as regards too large or too small congestion values. It is, however, clear that there are systematical deviations from the correct congestion values, since the hypotheses are approximate. For the time being, it can only be concluded that, for the studied cases, the calculated congestion values will generally give congestion values within the range $\pm$ 10 % of the correct ones.

Conclusions

From the comparisons made in chapter 3 it is seen that the dependence effect can be described with a relatively good precision by the methods presented in chapter 2. The method HS outlined in section 2.4 describes the dependence effect as a mutual deformation of the distributions in the B-columns and in the outlet columns. For arriving at correct traffic values, compared with the measurements, the method uses condition (2.2.22) which states that the traffic carried in the B-stage and in the outlets should be the same. For application of the method it is only necessary to use the traffic intensity values α_i for the inlets and the call destination factors η_{ij} to arrive at a result. Since it is not necessary to define the traffic offered to the routes, one big source of error for link system calculations is avoided.

It follows that the modified method HM, described in section 2.5, which works on the same principles as HS I and HS II as regards the deformation of the distributions, will vary in accuracy according to the correctness of the available values of the traffic offered to the routes. This is so because method HM does not satisfy condition (2.2.22).

From the comparisons in chapter 3 it is also seen that the new methods, which take into account the dependence effect, show a very clear improvement in accuracy compared with the values calculated under the independence assumption, HI. This is specially apparent when the congestion is large.

The new methods presented here were, however, not intended to be used for the practical dimensioning of link systems. They were only deduced for checking by means of measurements whether the analytical explanation of the dependence effect could cover the major part of the effects when the congestion is large. From the comparisons in chapter 3 it is seen that the analytically described deformation of the distributions agrees very well with measured distributions. There are, however, still small differences between calculated and measured

distributions, which will be discussed in the sequel. It can, however, already be concluded that the analytical description of the dependence effect in a link system is described in a way which gives the principal part of the truth. The thoughts behind the analytical explanation can consequently be used for building up a practical dimensioning method for link systems when the congestion is large.

It remains to explain the existing small differences between calculated and measured values. They can be explained by the fact that the calculated values follow the condition (2.2.22), while the measurements follow the more severe condition (1.3). The latter condition states that the total number of occupations is always the same in B-stage and outlets, while the former only states that it should be so on an average. It is consequently possible to select from the calculated distributions a number of combinations of p_{ij}, q_i, r_j, s_{ij} which cannot exist at the same time in accordance with condition (1.3). The probability for certain values of p_{ij}, q_i, r_j, s_{ij} can consequently be expected to be smaller than given by the calculations. Or, to say the same thing in another way, it can be expected that the measured distributions are a little more deformed than the calculated ones. Whether or not this is so, cannot be verified by the measurements. A study of the tables for the distributions in chapter 3 can neither confirm nor contradict this statement, since there are measured distributions that are more deformed than the calculated ones and others that are less deformed. Since the differences between calculated and measured distributions are small, the measurements are not accurate enough to permit a study of this secondary dependence effect. Such a study would have required considerably more calls per measurement.

The study relates to a link system in which all routes have large congestion values. It has been limited to such a case for two reasons. The first is that the deformations are more apparent in such a case than for a link system having routes with large congestion as well as routes with small congestion. The second reason was that the studied case is less complicated than the other, which refers to a group selector having both direct routes and overflow routes in the same multiple. There is, however, reason to discuss this latter case, which frequently occurs in practice, in the light of the results of the studies made in this paper.

The deformation of the distributions is explained by the congestion. The deformation of the distribution in a B-column is caused by the fact that a number of calls cannot occupy free B-links due to congestion on the called route. In the same way the deformation of the distribution for an outlet column is caused by blocked calls which cannot occupy free outlets due to lack of free links. Now, if a call can try a number of alternative outlet columns before it is rejected, it is of course more probable that any free link in the B-column can be used. The distribution for the B-column can consequently be expected to be less deformed than if there were no alternative routes. In fact the B-column can only be expected to have a deformed distribution in proportion to the resulting congestion for all calls from the B-column. If the resulting congestion is low, the deformation can be expected to be very little or negligible. It can be expected that the remaining deformations in the B-columns are mainly of the secondary type following from condition (1.3). On the other hand, a

practically undeformed distribution in the B-columns can be expected to cause more deformation of the distributions in the outlet columns than if the distributions for the B-columns were more seriously deformed. It follows that the calculation of the congestion for the direct routes is simpler in this case than for the one treated in this paper. This does not mean that the whole calculation is simpler, since the very difficult problem remains of calculating the congestion for the overflow routes, where the overflow traffic is composed of traffic rejected from a number of direct routes having deformed traffic distributions. Theoretical studies which take into account the deformations of the distributions for the direct routes have already been made by WALLSTRÖM (1961) and SMITH (1961). It is to be hoped that their studies will lead to a relatively simple and practical dimensioning method with sufficient accuracy for link systems employing alternative routing.

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Alternative Routing in a Two-Stage Link System. Congestion Theory and Simulated Traffic Studies

BY

BENGT WALLSTRÖM*

		page
CHAPTER 1	*Introduction*	262
CHAPTER 2	*A Theoretical Model*	263
	2.1 Assumptions	263
	2.2 Derivation of the State Distribution and Congestion on a Direct Route	264
	2.3 Derivation of Moments of Overflow Traffic	268
CHAPTER 3	*Measuring Facilities in the Traffic Simulations*	270
CHAPTER 4	*Comparisons with Simulated Traffic Studies*	273
	4.1 Survey of the Measurements	273
	4.2 Comparisons for Direct Routes	275
	4.3 Comparisons for Alternative Routes	282
CHAPTER 5	Conclusion	283
ACKNOWLEDGEMENTS		284
BIBLIOGRAPHY		284

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A method is proposed for calculation of the congestion on direct routes and the moments of overflow traffic in a two-stage link system with alternative routing. It offers the means in principle of applying Wilkinson's "Equivalent Random Theory" also to link systems. The theory is compared with results of traffic simulations which have been carried out on a digital computer for the group selector scheme of L M Ericsson's crossbar system ARF 10. Though the theory is derived for an infinitely large group selector stage, the agreement with measurements on finite systems is good, particularly for high congestion values.

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CHAPTER 1

Introduction

At the L M Ericsson Traffic Research Department studies are now being made of alternative routing arrangements in a link system. For this purpose an electronic computer is used to simulate traffic in a two-stage group selector. Earlier traffic trials (Wallström 1958) have shown that the congestion formulas given by Jacobaeus (1948) are sufficiently accurate for engineering purposes at the low rates of congestion generally prevailing on ordinary routes, though they generally overestimate the congestion a little. However, it has also appeared that this method is less accurate at the high congestion values occurring in alternative routing systems. In these cases the overestimation of traffic overflowing from the direct (or first choice) routes can be rather considerable. The traffic simulations have also shown that the overestimation of the congestion is principally due to the fact that the state distributions in the link system will be more favourable when the congestion increases than the full availability distributions used by Jacobaeus for low congestion conditions.

In engineering an alternative routing system the aim should be to attain an optimal division of traffic on direct and alternative routes. For this purpose we need formulas with sufficient accuracy also at high values of congestion. In chapter 2 of the present paper new formulas are derived for the congestion on a direct route and for the moments of traffic overflowing from a direct route. Thus it should be possible to apply the well-known "Equivalent Random Theory" suggested by R I Wilkinson (1956) also to link systems with alternative routing.

The new method is principally based on ideas that have been developed by A. Elldin in a paper in this issue, "On the dependence between the two stages in a link system" (Elldin 1961). A recent paper by Bretschneider (1961) gives a similar treatment of the problem. Cf. also Basharin (1960). However, the assumptions have been modified here in order to take into account the effects of overflow routes, which has not been done in the above-mentioned papers. Thus it is assumed that the effective congestion in the system is negligible, *i.e.* the alternative routes and routes without a second choice possibility can handle almost all calls offered to them. The consequence of this is that the traffic distribution in the B-stage will be unaffected by internal congestion. Moreover it is assumed that the number of inlet and outlet columns is infinite (cf. Jacobaeus 1950). As a result of these assumptions rather simple formulas have been arrived at. The congestion and moments are obtained from recurrences by straight forward computation.

In chapter 3 the measuring facilities of the computer programme are summed up. Systems with up to 30 inlet columns and 20 outlet columns can be handled by the computer. During a simulation eight different congestion values and seven state distributions are measured.

In chapter 4 comparisons are made between the new theory and simulated traffic studies on the group selector scheme of L M Ericsson's crossbar system *ARF 10.*

CHAPTER 2

A Theoretical Model

2.1 Assumptions

The type of link system to be studied is shown in *Fig. 1.* The inlets are indicated by A, the links by B, and the outlets by C. There are k columns of inlets and links, each column having n inlets and m links. The outlets are divided between direct routes, alternative routes, and ordinary routes without a second choice possibility. The present study will be limited to direct routes consisting of one C-column. The following assumptions are made:

1. The inlets of the system are calling independently of one another, all with equal intensities. The idle times of an inlet are exponentially distributed with mean $1:y$, where y is the call intensity of a free inlet. An occupied inlet has zero call intensity.
2. The calls are distributed at random among the routes and there is a constant probability η of an arbitrary call being directed to the route to be studied.
3. The holding times are exponentially distributed with the same mean value, s, for all types of calls.

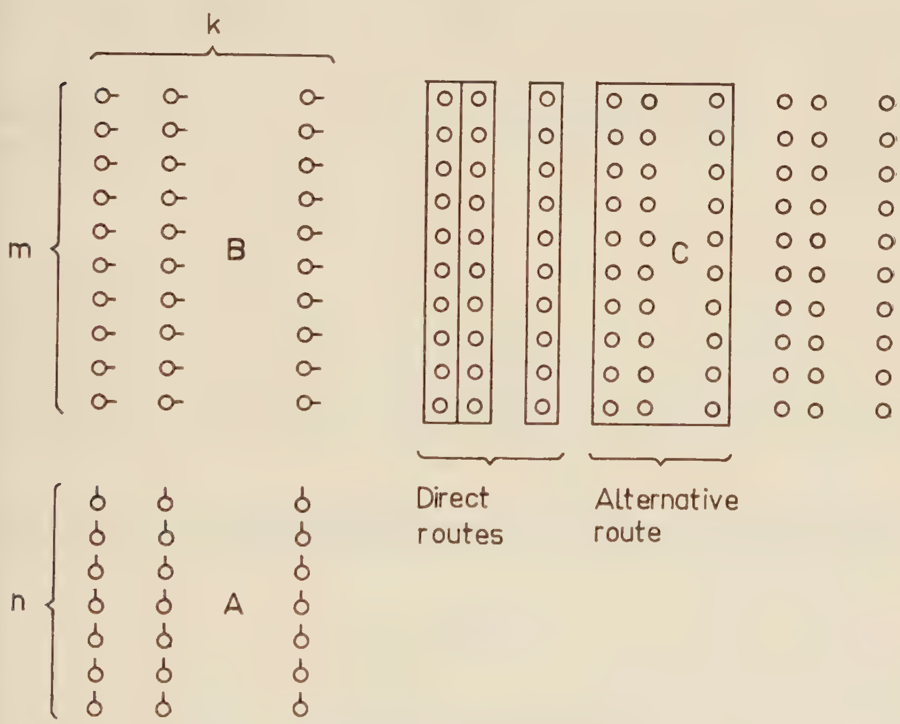

Fig. 1. Two-stage group selector with direct and alternative routes.

4. The number of inlet columns, k, is infinite, as also is the number of outlet columns (cf. JACOBAEUS 1950). The probability η defined in item 2 above is infinitesimally small.
5. The congestion on alternative routes and ordinary routes is zero.
6. $n \leq m$
7. Random hunting for free and accessible outlets in a C-column is used.

The first of these assumptions is perhaps a little dubious because in reality the inlets do not act as completely independent traffic sources. The inlets of a group selector stage are generally divided between several incoming routes, and the circuits of one route are connected as far as possible to different inlet columns. In a system, for instance, in which each incoming route has k circuits forming a horisontal row in the A-stage, all inlets of a column will clearly belong to different incoming routes. In this case the inlets of a column can be regarded as independent sources. Consequently, if the load per circuit is the same on all the incoming routes, the states of occupation in an inlet column will follow the Bernouilli distribution. This will also be the case according to the assumptions above. The states in different columns will, however, not be completely independent in practice. It can also be concluded that, if the incoming routes do have not equal loads per circuit, the distribution in an inlet column will have less variance than the Bernouilli distribution. However, the opposite can probably also occur if the incoming routes consist of several horisontal inlet rows.

It may be concluded from the above argument that there are rather good reasons to assume the Bernouilli distribution on an inlet column irrespective of the distributions prevailing on the incoming routes. However, assumption 1 was also chosen with the aim of simplifying the theory and the traffic simulations. But this assumption implies that the total traffic on the $n \cdot k$ inlets will be Bernouilli-distributed, which may not be expected to occur in practice. The consequences of this will be discussed in connection with the measurements on alternative routes in chapter 4.

2.2 Derivation of the State Distribution and Congestion on a Direct Route

Fig. 2 shows a column with n inlets and m links together with a direct route of m outlets. Occupied devices are shown black and it is seen that congestion exists between the inlets and outlets.

According to assumption 5, no calls will be rejected from the inlets. The states of occupation in the B-column will consequently follow the Bernouilli distribution

$$[p]_B = \binom{n}{p} \cdot a^p \cdot (1-a)^{n-p}$$

where

$$a = \frac{sy}{1+sy}$$

is the average load on an inlet.

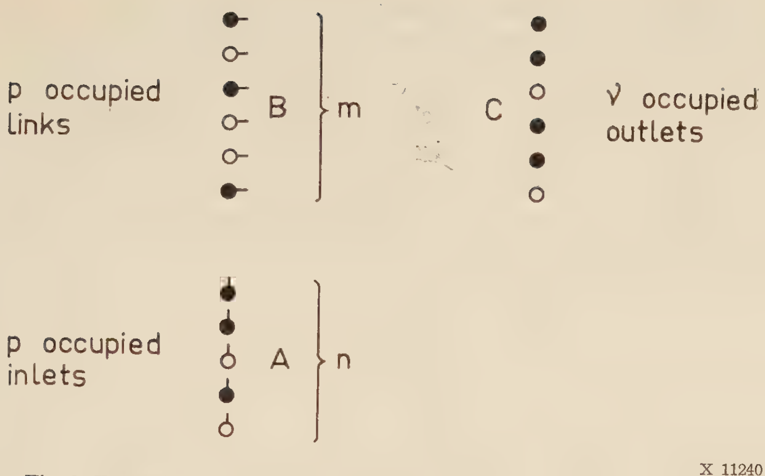

Fig. 2. Example of occupations in one AB-column and one direct route.

The average number of calls per unit of time occurring at state p in the column is

$$(n-p)\cdot y\cdot [p]_B$$

while the total number of calls per unit of time is

$$\sum_{p=0}^{n} (n-p)\cdot y\cdot [p]_B = n\cdot y\cdot (1-a)$$

The ratio of these two expressions,

$$G_{(p)} = \frac{(n-p)\cdot [p]_B}{n\cdot (1-a)} = \binom{n-1}{p}\cdot a^p\cdot (1-a)^{n-1-p} \tag{2.2.1}$$

equals the probability that a call selected at random has occurred at state p in a B-column.

Consider now the traffic offered to the direct route by all $n\cdot k$ inlets of the system. The probability of altogether r occupations existing in this traffic, either on the direct or on the alternative route, is

$$[r]_k = \binom{nk}{r}\cdot (\eta a)^r\cdot (1-\eta a)^{nk-r}$$

It is well known that, when k tends to infinity and η tends to zero while $n\cdot k\cdot \eta\cdot a = A$ remains constant, the above Bernouilli-distribution will asymptotically approach the Poissonian distribution

$$[r] = \frac{A^r}{r!}\cdot e^{-A}$$

Further it is realized that the average time interval between successive calls from a specified column to the route considered will tend to infinity. The state in the B-column when a call to the route occurs will thus tend to be independent of the state prevailing when the last

preceding call from the column to the route occurred. Consequently the probability (2.2.1) of a randomly selected call occurring at state p in a B-column will be valid for all calls to the route.

It is also clear that the states on a B-column will tend to be independent of the states on the route considered. Since random hunting has been assumed, the occupations in a B-column must be considered as distributed at random among the devices, and the same holds for the route. The probability of a call being blocked on the condition that altogether ν outlets on the route are occupied will then be

$$H_\nu = \sum_{p=m-\nu}^{n-1} G(p) \cdot \frac{\binom{\nu}{m-p}}{\binom{m}{p}}$$

Using (2.2.1) for $G(p)$ the sum is reduced to

$$\left.\begin{aligned} H_\nu &= \frac{\binom{n-1}{m-\nu}}{\binom{m}{m-\nu}} \cdot a^{m-\nu} \qquad &&\text{for } \nu = m-n+1, \ldots, m \\ H_\nu &\equiv 0 &&\text{for } \nu = 0,1, \ldots, m-n \end{aligned}\right\} \tag{2.2.2}$$

The probability of a call not being blocked equals $1 - H_\nu$.

Since it has been found that the calls to the route will follow a Poisson process, the following equilibrium equations are obtained for the transitions between states on the route:

$$\{A \cdot (1 - H_\nu) + \nu\} \cdot [\nu] = (\nu + 1) \cdot [\nu + 1] + A \cdot (1 - H_{\nu-1}) \cdot [\nu - 1] \tag{2.2.3}$$

where $[\nu]$, $(\nu = 0,1, \ldots, m)$ are the probabilities of states on the route and A is the traffic offered to the route.

One has $[m + 1] \equiv 0$ and $[-1] \equiv 0$.

It is found that the system is satisfied by the more convenient relation

$$\nu \cdot [\nu] = A \cdot (1 - H_{\nu-1}) \cdot [\nu - 1]$$
$$(\nu = 1,2, \ldots m) \tag{2.2.4}$$

from which the solution is found by iterations.

$$[\nu] = \frac{A^\nu}{\nu!} \cdot \prod_{s=0}^{\nu-1} (1 - H_s) \cdot [0] \tag{2.2.5}$$

where [0] is determined by the condition

$$\sum_{\nu=0}^{m} [\nu] = 1 \tag{2.2.6}$$

It is seen from (2.2.2) that H_s tends to zero as a tends to zero. The expression (2.2.5) will then approach the Erlang distribution for a full availability group. The solution (2.2.5) can thus be regarded as a deformed Erlang distribution (cf. ELLDIN 1961). The deformation is governed by the factor

$$\prod_{s=0}^{\nu-1} (1 - H_s)$$

which is unity for $\nu = 1,2,\ldots, m - n + 1$ and monotonously decreasing for $\nu = m - n + 2, \ldots, m$.

The call congestion, B, to a direct route can be calculated as the ratio of traffic blocked to traffic offered,

$$B = \frac{A - \sum_{\nu=1}^{m} \nu \cdot [\nu]}{A} \qquad (2.2.7)$$

Using (2.2.4) in (2.2.7) we obtain alternatively

$$B = \sum_{\nu=0}^{m} H_\nu \cdot [\nu] \qquad (2.2.8)$$

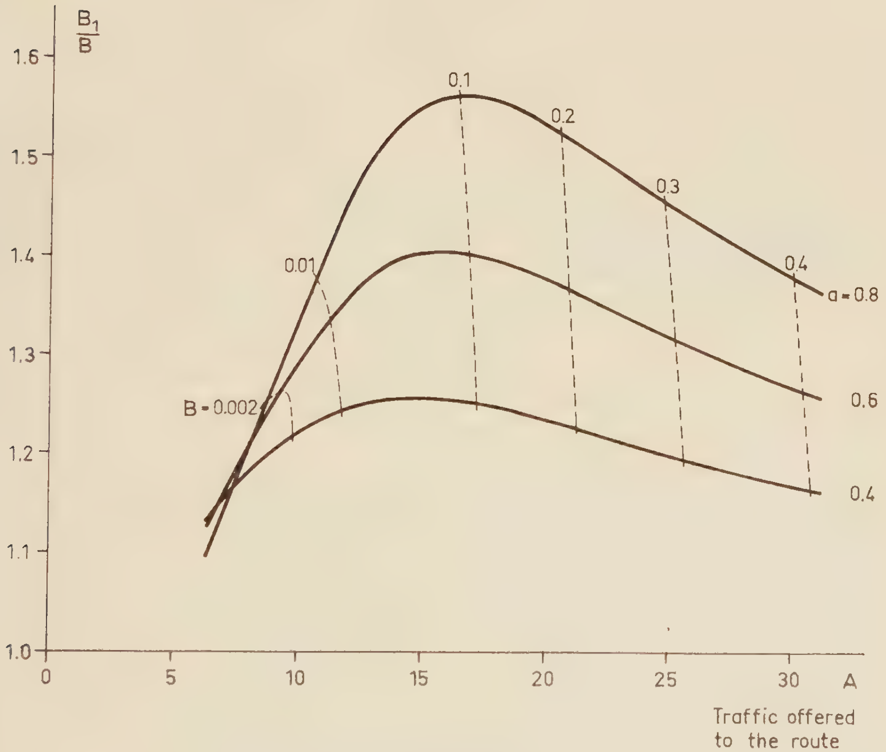

Fig. 3. Comparison between B (2.2.8) and B_1 (2.2.9).

The latter expression is formally identical to the basic congestion formula given by Jacobaeus. Thus if $[\nu]$ in (2.2.8) is replaced by the Erlang distribution

$$[\nu]_1 = \frac{\frac{A^\nu}{\nu!}}{1 + A + \cdot \cdot \cdot \cdot \cdot + \frac{A^m}{m}},$$

the wellknown formula

$$B_1 = \frac{E_m(A)}{E_{n-1}\left(\frac{A}{a}\right)} \tag{2.2.9}$$

is obtained.

The formulas (2.2.8) and (2.2.9) contain the same parameters, so may be easily compared *Fig. 3* shows how the ratio of (2.2.9) to (2.2.8) varies with traffic offered on a route of 20 devices in L M Ericsson's crossbar system ARF 10 ($m = 20$, $n = 13$) for the inlet loads $a = 0.4$, 0.6 and 0.8. It is seen that formula (2.2.9) gives considerably higher values than (2.2.8) for high values of congestion and inlet load.

2.3 Derivation of moments of overflow traffic

Let $[\nu, \mu]$ be the probability that at a random instant ν calls exist on a certain direct route and μ calls which have been rejected from that route exist on the alternative route. On the assumptions given in (2.1) and using the probability H_ν derived in (2.2) we obtain the following equilibrium equations:

$$\begin{aligned}(A + \nu + \mu) \cdot [\nu, \mu] = {} & (\nu + 1) \cdot [\nu + 1, \mu] \\ & + (\mu + 1) \cdot [\nu, \mu + 1] \\ & + A \cdot (1 - H_{\nu - 1}) \cdot [\nu - 1, \mu] \\ & + A \cdot H_\nu \cdot [\nu, \mu - 1]\end{aligned} \tag{2.3.1}$$

$\nu = 0,1,\ldots, m$

$\mu = 0,1,\ldots, \infty$ (cf. assumption 4)

One has $[m+1,\mu] \equiv 0$, $[-1,\mu] \equiv 0$ and $[\nu,-1] \equiv 0$

With ν fixed, factorial moments of μ are defined by

$$M_h(\nu) = \begin{cases} \sum_{\mu=0}^{\infty} [\nu, \mu] = [\nu] & \text{for } h = 0 \\ \sum_{\mu=0}^{\infty} \mu \cdot (\mu - 1) \cdot \cdot \cdot (\mu - h + 1)[\nu, \mu] & \text{for } h = 1,2,\ldots \end{cases} \tag{2.3.2}$$

These moments will be derived following a method outlined by J. Riordan (WILKINSON 1956) for full availability conditions. Accordingly we introduce the factorial moment generating function

$$M(\nu, t) = \sum_{h=0}^{\infty} M_h(\nu) \cdot \frac{t^h}{h!} = \sum_{\mu=0}^{\infty} [\nu, \mu] \cdot (1+t)^{\mu} \tag{2.3.3}$$

Multiplying (2.3.1) by $(1+t)^{\mu}$ and summing for μ we obtain, using (2.3.3),

$$\begin{aligned}
&(A+\nu) \cdot M(\nu, t) + (1+t) \cdot \frac{d}{dt}\left\{ M(\nu, t) \right\} \\
&= (\nu+1) \cdot M(\nu+1, t) + \frac{d}{dt}\left\{ M(\nu, t) \right\} \\
&+ A(1 - H_{\nu-1}) \cdot M(\nu-1, t) \\
&+ A \cdot H_\nu \cdot (1+t) \cdot M(\nu, t)
\end{aligned} \tag{2.3.4}$$

Equation (2.3.4) must hold for arbitrary values of t. Consequently the coefficients for all powers of t must be identically zero. Using this condition the following recurrence is found:

$$\begin{aligned}
(\nu+1) \cdot M_h(\nu+1) = M_h(\nu) &\cdot \{\nu + h + A \cdot (1 - H_\nu)\} \\
&- M_h(\nu-1) \cdot A(1 - H_{\nu-1}) \\
&- M_{h-1}(\nu) \cdot A \cdot h \cdot H_\nu
\end{aligned} \tag{2.3.5}$$

$(\nu = 0, 1, \ldots, m-1)$

$M_h(-1) \equiv 0$

At this point we have to leave Riordan's method of derivation, because it does not seem feasible to find explicit solutions. In the following an additional formula will be derived, which can be used together with (2.3.5) for calculation of the moments. It can easily be verified that (2.3.1) is satisfied by the following relation:

$$\sum_{\nu=0}^{m} \mu \cdot [\nu, \mu] = \sum_{\nu=0}^{m} A \cdot H_\nu \cdot [\nu, \mu - 1] \tag{2.3.6}$$

This equation points out that the number of transitions from state μ to state $\mu - 1$ in the overflow traffic will on average equal the number of transitions from state $\mu - 1$ to state μ.

Multiplying both members of (2.3.6) by

$(\mu - 1) \cdot (\mu - 2) \cdot \cdot \cdot (\mu - h + 1)$

and summing for μ, we obtain

$$M_h = \sum_{\nu=0}^{m} M_h(\nu) = A \cdot \sum_{\nu=0}^{m} H_\nu \cdot M_{h-1}(\nu) \tag{2.3.7}$$

Starting with the mean value

$$E(\mu) = M_1 = A \cdot \sum_{\nu=0}^{m} H_\nu \cdot [\nu] = A \cdot B \tag{2.3.8}$$

it is now possible successively to calculate moments of higher order. The variance is expressed by

$$D^2(\mu) = M_1 - M_1^2 + M_2 = A \cdot B \cdot \left(1 - A \cdot B + \frac{1}{B} \cdot \sum_{\nu=0}^{m} H_\nu \cdot M_1(\nu)\right) \tag{2.3.9}$$

The $M_1(\nu)$ are expressed by (2.3.5) as linear functions of $M_1(0)$ and of $M_0(\nu)$, which are equal to $[\nu]$ as given by (2.2.5). From (2.3.7) we obtain

$$\sum_{\nu=0}^{m} M_1(\nu) = A \cdot B \tag{2.3.10}$$

by which all $M_1(\nu)$ are determined. Using these values in (2.3.5) and (2.3.7) the $M_2(\nu)$ and M_3 can be calculated, and so on.

Obviously this procedure can be used for all kinds of direct routes — graded or not — provided that the factors H_ν can be calculated for a number of typical cases. Also other switching arrangements with restricted availability may be treated in the same way.

According to Wilkinson's "Equivalent Random Theory" the direct routes are replaced by a fictitious group with Poisson input. The number of devices in this group and the traffic offered are determined by the condition that the mean and variance of the total traffic on the alternative route must remain unchanged. The variance is calculated as the sum of variances of the traffic streams overflowing from the individual direct routes. The procedure is based on the assumption that the traffic emanates from an infinite number of sources and follows the Poissonian distribution. The addition law for variances should be valid in the link system, too, if the number of inlet columns is sufficiently large. Therefore it seems probable that Wilkinson's method or some similar procedure can be used in estimating the grade of service on the alternative routes.

CHAPTER 3

Measuring Facilities in the Traffic Simulations

The traffic simulations are carried out as Markoff chain experiments using uniformly distributed random numbers. This method has been described by, for instance, JENSEN (1952_{II}, 1953), KOSTEN (1955), and NEOVIUS (1955). In a paper by the author (WALLSTRÖM 1958) the application of the method to a two-stage link system has been dealt with, as well as the computer programming.

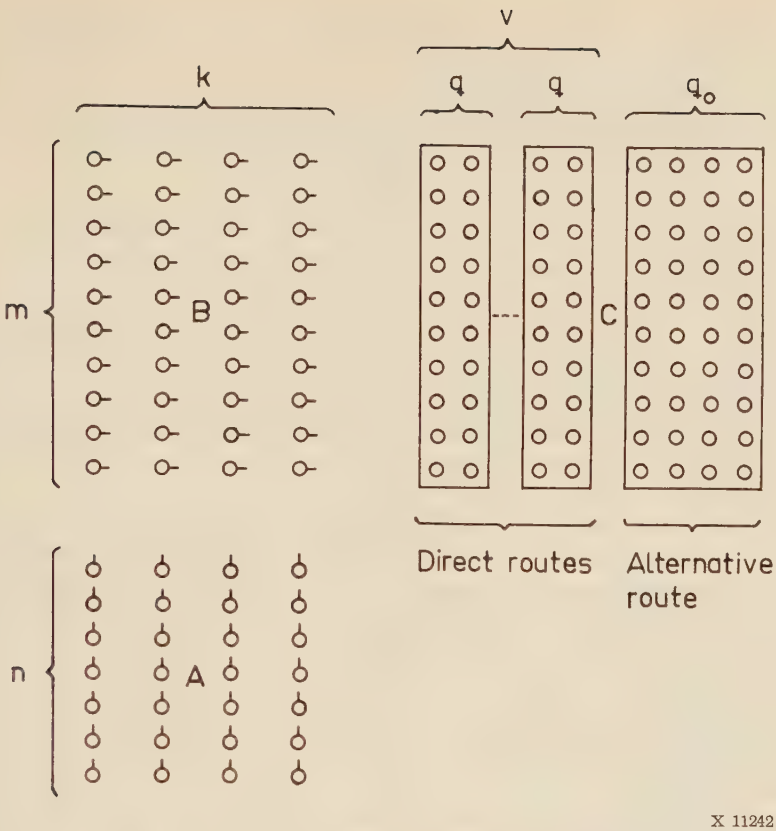

Fig. 4. Group selector scheme for simulated traffic studies.

The simulations related in the present paper have been performed on link systems of the type shown in *Fig. 4* with various numbers of direct routes but with only one alternative route. The signification of the symbols in *Fig. 4* will be clear from the list below of input data for the computer programme. The limitations placed on the size of the system are determined by the memory capacity of the computer.

k Number of columns in the A- and B-stages. ($k \leq 30$)

n Number of inlets per column.

m Number of links per B-column and number of outlets per C-column. ($m \leq 40$)

v Number of direct routes.

q Number of columns forming each direct route.

q_0 Number of columns forming the alternative route. ($\mathrm{v} \cdot q + q_0 \leq 20$)

s_1 If $s_1 = 0$ the columns of a route are tested in the order 1, 2, . . . , q. If $s_1 = 1$ they are tested cyclically, starting on a column chosen at random.

s_2 If $s_2 = 0$ the free and accessible outlets of a column are hunted in the order 1, 2, . . . , m. If $s_2 = 1$ random hunting is performed.

α_i Traffic offered by a free inlet in the i:th column ($i = 1,2, \ldots, k$). A free inlet initiates calls with constant intensity while an occupied inlet has zero call intensity. The holding times follow the exponential law.

η_j The probability of an arbitrary call being directed to the j:th direct route ($j = 1, 2, \ldots, v$).

If
$$\sum_{j=1}^{v} \eta_j < 1 \tag{3.2}$$
the proportion
$$\eta_o = 1 - \sum_{j=1}^{v} \eta_j$$
of the calls will be directed to the alternative route. Calls rejected from the direct routes are also offered to the alternative route.

γ Traffic distributions in the system are estimated by sampling with exponential time intervals and an intensity of γ samples per average holding time.

$\left.\begin{matrix} N_0 \\ N \\ X \end{matrix}\right\}$ A simulation is started with N_0 calls to bring the traffic process into statistical equilibrium. After that a measurement comprising $X \cdot N$ calls is made and finally the results are printed. From the X call series, estimates of the standard deviations of the congestion values are computed.

The following items are measured during a simulation.

1. Call congestion
   a) from all A-columns to all direct routes
   b) from the first A-column to all direct routes
   c) from all A-columns to the first direct route
   d) from the first A-column to the first direct route
   e) from all A-columns to the alternative route
   f) for all overflow calls to the alternative route
2. Time congestion
   a) from the first A-column to the first direct route
   b) from the first A-column to the alternative route
3. Distributions
   a) for the number of occupied links in the first B-column
   b) for the number of busy q-tuples [1] in the first direct route
   c) for the number of busy BC-pairs[2] in the first B-column and the first direct route
   d) for the number of busy q_0-tuples in the alternative route
   e) for the number of busy BC-pairs in the first B-column and the alternative route
   f) for the total number of occupied outlets in the alternative route
   g) for the number of simultaneous occupations in the alternative route that have been rejected from the first direct route.

[1] By a "busy q-tuple" is meant that the q outlets in a horisontal row of the route are all occupied.
[2] A "BC-pair" consists of a link and the corresponding q-tuple. The BC-pair is busy if one or both of the link and q-tuple is busy.

Comparisons with Simulated Traffic Studies

4.1 Survey of the Measurements

In this chapter comparisons will be made between measurements and theory for the group selector arrangement in L M Ericsson's crossbar system ARF 10. It is a two-stage link system of the type shown in *Fig. 1* with $n = 13$ inlets per A-column and $m = 20$ outlets per C-column. A survey of the measurements made is given in *Table 1.*

The traffic simulations were planned to conform as closely as possible with the assumptions in chapter 2.1. Thus the call intensities of free inlets were put equal for all columns, *i.e.*

$$\alpha_1 = \alpha_2 = \cdot \cdot \cdot = \alpha_k = \alpha$$

Table 1. Survey of traffic trials.

Trial no.	k	v	m	n	q	q_o	s_1	s_2	α	η	γ	N_o	N	X
3,046	12	3	20	13	1	8	1	1	1.5	0.333333	20	2,000	1,000	10
3,047	16	4	20	13	1	8	1	1	1.5	0.25	20	2,000	1,000	10
3,048	20	5	20	13	1	8	1	1	1.5	0.2	20	2,000	1,000	10
3,049	24	6	20	13	1	8	1	1	1.5	0.166667	20	2,000	1,000	10
3,050	28	7	20	13	1	8	1	1	1.5	0.142857	20	2,000	1,000	10
3,051	8	4	20	13	1	8	1	1	4	0.25	20	2,000	1,000	10
3,052	10	5	20	13	1	8	1	1	4	0.2	20	2,000	1,000	10
3,053	14	7	20	13	1	8	1	1	4	0.142857	20	2,000	1,000	10
3,054	16	8	20	13	1	8	1	1	4	0.125	20	2,000	1,000	10
3,055	20	10	20	13	1	8	1	1	4	0.1	20	2,000	1,000	10
3,056	28	10	20	13	1	1	1	1	0.666667	0.1	40	2,000	1,000	20
3,057	27	10	20	13	1	1	1	1	0.666667	0.1	40	5,000	5,000	20
3,058	18	10	20	13	1	1	1	1	1.5	0.1	40	5,000	5,000	20
3,059	13	10	20	13	1	1	1	1	4	0.1	40	5,000	5,000	20
3,060	20	5	20	13	1	8	1	1	1.5	0.2	40	2,000	2,500	20
3,061	20	5	20	13	1	8	1	1	1.5	0.16	40	2,000	2,500	20
3,062	20	5	20	13	1	8	1	1	1.5	0.135	40	2,000	2,500	20
3,063	20	5	20	13	1	8	1	1	1.5	0.11	40	2,000	2,500	20
3,064	20	5	20	13	1	8	1	1	1.5	0.08	40	2,000	2,500	20
3,065	15	5	20	13	1	8	1	1	4	0.2	50	5,000	2,500	20
3,066	15	5	20	13	1	8	1	1	4	0.135	50	5,000	2,500	20
3,067	20	5	20	13	1	8	1	1	1.5	0.2	50	5,000	2,500	20
3,068	20	5	20	13	1	8	1	1	1.5	0.135	50	5,000	2,500	20
3,069	30	5	20	13	1	8	1	1	0.666667	0.2	50	5,000	2,500	20
3,070	30	5	20	13	1	8	1	1	0.666667	0.135	50	5,000	2,500	20
3,071	30	10	20	13	1	12	1	1	4	0.1	50	5,000	2,500	20
3,072	30	10	20	13	1	12	1	1	4	0.0675	50	5,000	2,500	20
3,073	30	5	20	13	1	8	1	1	0.666667	0.2	50	5,000	2,500	20
3,074	20	5	20	13	1	8	1	1	1.5	0.2	50	5,000	2,500	20
3,075	30	10	20	13	1	10	1	1	4	0.1	50	5,000	2,500	20
3,076	21	5	20	13	1	5	1	1	1.5	0.16	40	5,000	2,500	20
3,077	20	5	20	13	1	8	1	1	1.5	0.08	50	5,000	5,000	20

The direct routes, each consisting of one C-column, were offered equal traffic intensities in order to favour the congestion estimate, *i.e.*

$$\eta_1 = \eta_2 = \cdots = \eta_v = \eta$$

The alternative route was in all trials designed for practically zero congestion. Thus the traffic offered by an inlet can be calculated as

$$a = \frac{\alpha}{1+\alpha}$$

and the traffic offered to a direct route as

$$A = \eta \cdot n \cdot k \cdot a$$

Random hunting on the routes was performed both horisontally and vertically, *i.e.* $s_1 = s_2 = 1$ (see chapter 3).

In 2.1 the number of inlet and outlet columns were assumed to be infinite. Therefore a first series of trials, 3,046—3,055, each comprising 10,000 calls, were made in order to study the influence of these factors. The results are shown in *Table 2*, where B^* denotes the call congestion measured on direct routes.

Table 2. Measured call congestion on direct routes for various numbers of inlet and outlet columns.
$m = 20,\ n = 13,\ q = 1,\ v \cdot \eta = 1$

Trial no.	k	v	q_0	a	A	B^*	95 % confidence interval
3,046	12	3	8	0.6	31.2	0.4032	0.3928—0.4136
3,047	16	4	8	0.6	31.2	0.4064	0.3950—0.4178
3,048	20	5	8	0.6	31.2	0.4146	0.4064—0.4228
3,049	24	6	8	0.6	31.2	0.4198	0.4096—0.4300
3,050	28	7	8	0.6	31.2	0.4105	0.4007—0.4203
3,051	8	4	8	0.8	20.8	0.1851	0.1757—0.1945
3,052	10	5	8	0.8	20.8	0.1895	0.1825—0.1965
3,053	14	7	8	0.8	20.8	0.1979	0.1867—0.2091
3,054	16	8	8	0.8	20.8	0.1922	0.1838—0.2006
3,055	20	10	8	0.8	20.8	0.2034	0.1968—0.2100

It is seen that the increase of congestion with an increasing dimension of system, which has been demonstrated more clearly by earlier measurements (WALLSTRÖM 1958), seems rather unimportant for the larger values of k and $v + q_0$. The dimensions of systems that can be handled by the computer should consequently be sufficiently large for the purpose of checking the theory.

Trials nos. 3,056—3,077 will be used in the following for comparisons with the theory in chapter 2. It is seen that the input values for some of the trials are identical. The reason for this is that new measuring facilities have been added to the computer programme during the course of the investigations, wherefore some of the trials have been repeated. Thus the total state distribution on the alternative route and the overflow distribution from a single direct route have not been measured on all trials. On the other hand, different random numbers

have been used in otherwise identical simulations. They should consequently be regarded as independent measurements.

The number of calls N_0 for attaining equilibrium conditions is seen to be 2,000 or 5,000. The latter number was used in the trials in which distributions on the alternative route were measured because the overflow traffic was assumed to converge more slowly than random traffic towards statistical equilibrium. The simulations have been performed by the computer at a speed of about 40,000 calls an hour.

4.2 Comparisons for Direct Routes

In *Table 3* comparisons are made of congestion values B and B^* on direct routes, obtained from formula (2.2.8) and by measurements respectively. As previously, a is the load per inlet and A is the traffic offered to a direct route. The total number of calls, $X \cdot N$, per trial (see Table 1) is 50,000 except for a few cases with low congestion, which have been extended to 100,000 calls. The estimates of the standard deviation σ^* of B^*, have been computed from a sample of $X = 20$ observations in all cases. The interval $B^* \pm 2\sigma^*$ should contain the true congestion value with approximately 95 % confidence.

In *Fig. 5—7* the results of *Table 3* are represented in the form of ratios $B^*:B$ together with confidence intervals. Some of these have been obtained by combination of two or three measurements with equal values of a and A. For comparison two curves have been drawn, the upper one representing Jacobaeus' formula (2.2.9) and the lower one Erlang's formula $E_m(A)$, both divided by B.

Table 3. Comparison with measurements on direct routes.

$m = 20, \quad n = 13, \quad q = 1$

Trial no.	k	v	q_0	η	a	A	B	B^*	$B^* \pm 2\sigma^*$
3,069	30	5	8	0.2	0.4	31.2	0.4066	0.4044	0.3982—0.4106
3,073	30	5	8	0.2	0.4	31.2	0.4066	0.4069	0.4015—0.4123
3,070	30	5	8	0.135	0.4	21.06	0.1948	0.1950	0.1874—0.2026
3,057	27	10	1	0.1	0.4	14.04	0.0363	0.0346	0.0320—0.0372
3,060	20	5	8	0.2	0.6	31.2	0.4115	0.4128	0.4082—0.4174
3,067	20	5	8	0.2	0.6	31.2	0.4115	0.4121	0.4073—0.4169
3,074	20	5	8	0.2	0.6	31.2	0.4115	0.4126	0.4086—0.4166
3,061	20	5	8	0.16	0.6	24.96	0.2938	0.2878	0.2836—0.2920
3,062	20	5	8	0.135	0.6	21.06	0.2038	0.1998	0.1934—0.2062
3,068	20	5	8	0.135	0.6	21.06	0.2038	0.2012	0.1946—0.2078
3,063	20	5	8	0.11	0.6	17.16	0.1062	0.1005	0.0939—0.1071
3,064	20	5	8	0.08	0.6	12.48	0.0207	0.0198	0.0170—0.0226
3,077	20	5	8	0.08	0.6	12.48	0.0207	0.0184	0.0162—0.0207
3,058	18	10	1	0.1	0.6	14.04	0.0420	0.0364	0.0348—0.0380
3,065	15	5	8	0.2	0.8	31.2	0.4186	0.4111	0.4073—0.4149
3,075	30	10	10	0.1	0.8	31.2	0.4186	0.4171	0.4129—0.4213
3,066	15	5	8	0.135	0.8	21.06	0.2147	0.2021	0.1957—0.2085
3,072	30	10	12	0.0675	0.8	21.06	0.2147	0.2083	0.2029—0.2137
3,059	13	10	1	0.1	0.8	13.52	0.0424	0.0346	0.0332—0.0360

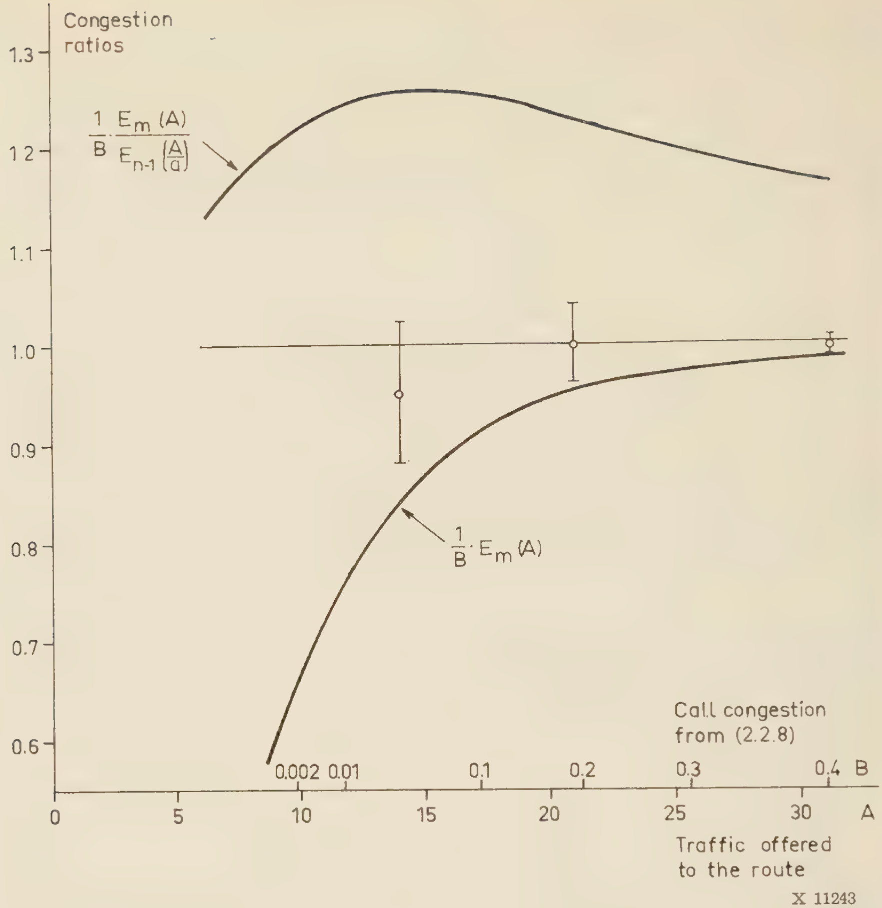

Fig. 5. Comparison of measured call congestion B^* with calculated call congestion B from formula (2.2.8).

Ratio $B^*:B$ with 95 % confidence interval.
$m = 20$, $n = 13$, $a = 0.4$

It is seen that the measured congestion values are generally a little lower than those calculated from (2.2.8). In comparison with formula (2.2.9), however, a considerable improvement of the accuracy has been achieved. The best agreement with measurements is observed for high congestion values. Since, as far as the author can judge, no mathematical approximations have been made in the derivation of (2.2.8), the disagreements observed particularly for the lower congestion values must be due to the limited number of columns in the systems studied by traffic simulations. In fact all other assumptions have been realized in the trials. Consequently one can expect to find the best agreement with measurements for the largest systems. This is also confirmed by the results (cf., e.g., trials nos. 3,057, 3,058 and 3,059, where $A = 14.04$ erl.). It may seem curious that the largest deviations from the theory

occur for the lower values of congestion. A probable explanation is that the deformation of distributions is the dominating factor when the congestion is high, while other effects become important when the congestion decreases, viz. (cf. ELLDIN 1961) dependence between the numbers of simultaneous occupations in the *B*- and *C*-columns, non-random occupation of the devices in the columns and non-pure Poissonian call processes offered to the routes. These effects will not prevail in an infinite system of the type studied in chapter 2.

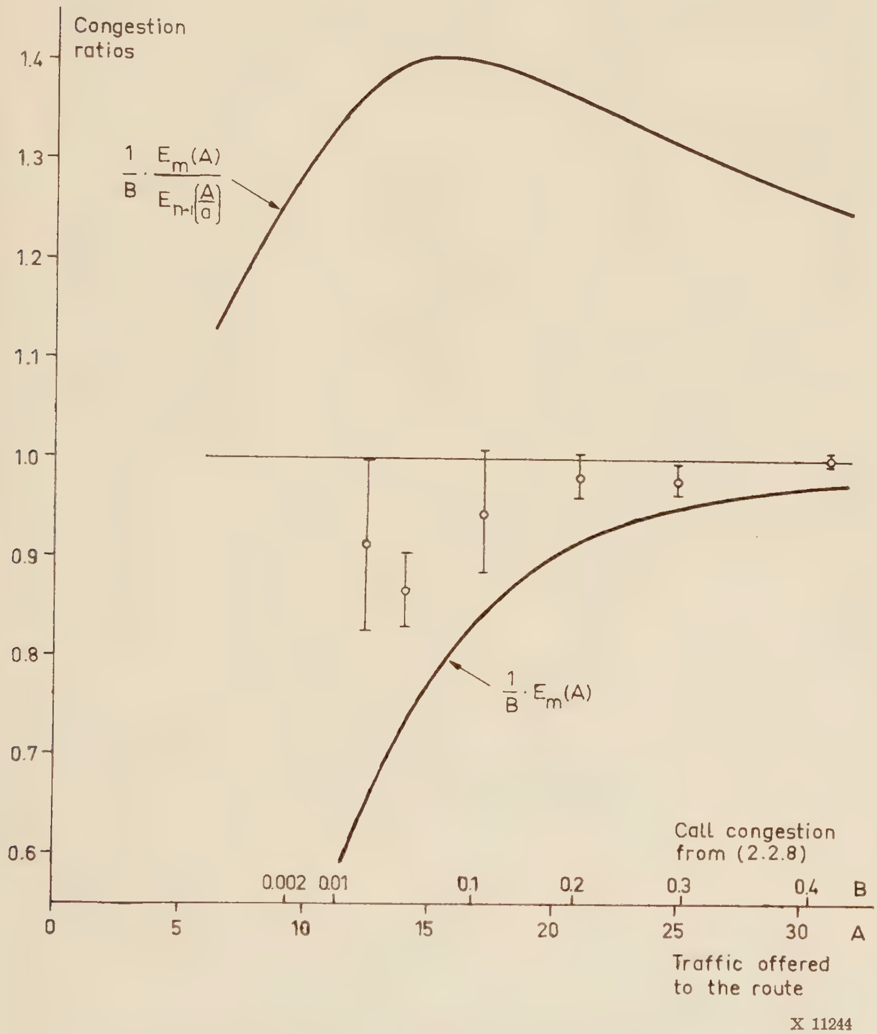

Fig. 6. Comparison of measured call congestion B^* with calculated call congestion B from formula (2.2.8).

Ratio $B^*:B$ with 95 % confidence interval.
$m = 20, \ n = 13, \ a = 0.6.$

9—106690

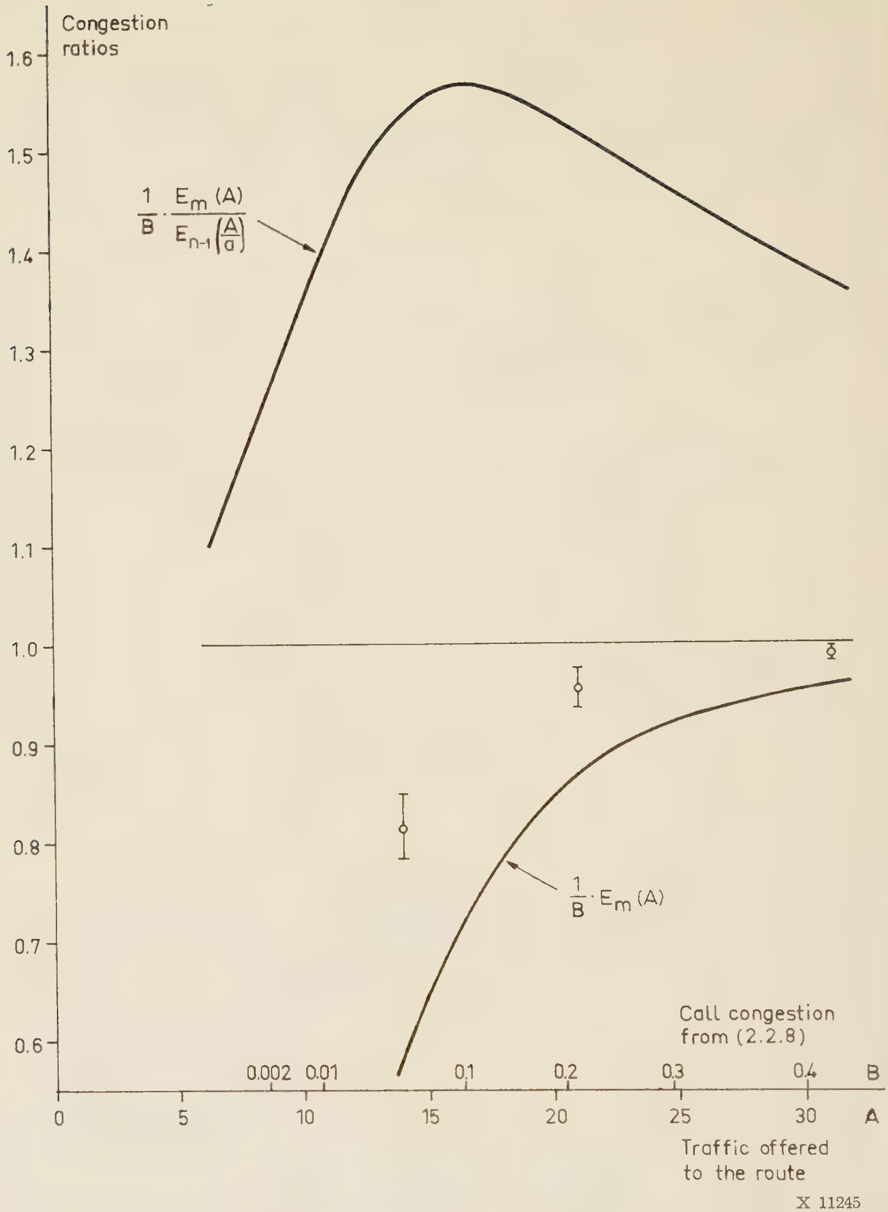

Fig. 7. Comparison of measured call congestion B^* with calculated call congestion B from formula (2.2.8).

Ratio $B^*:B$ with 95 % confidence interval.
$m - 20$, $n = 13$, $a = 0.8$.

In *Fig. 8*, *9* and *10* some measured state distributions in a direct route are compared with formula (2.2.5) and with the Erlang distribution. The traffic offered to the route is the same in the three cases ($A = 21.06$), while the inlet loads are different ($a = 0.4$, 0.6 and 0.8). The agreement between measured distributions and formula (2.2.5) is seen to be good. It is also observed that the Erlang distribution is clearly unsatisfactory, particularly when the load per inlet is high.

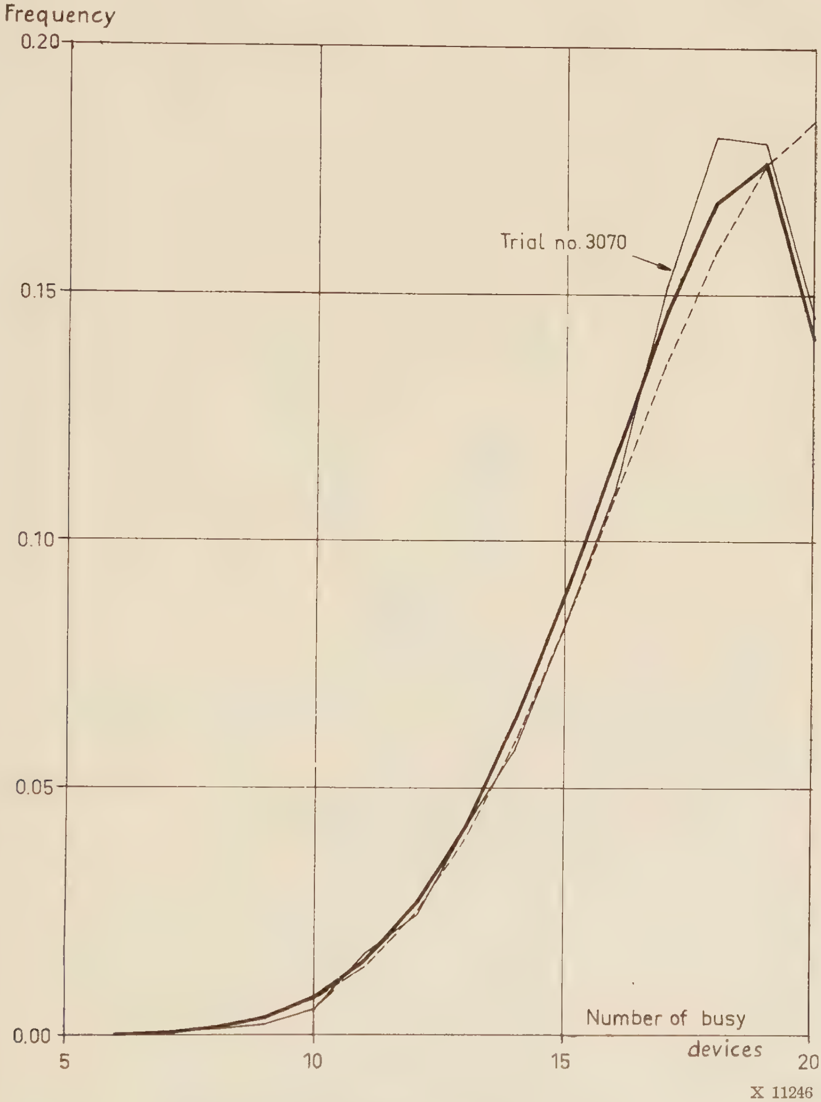

Fig. 8. State distribution on a direct route.
$A = 21.06$, $a = 0.4$.

9*—106690

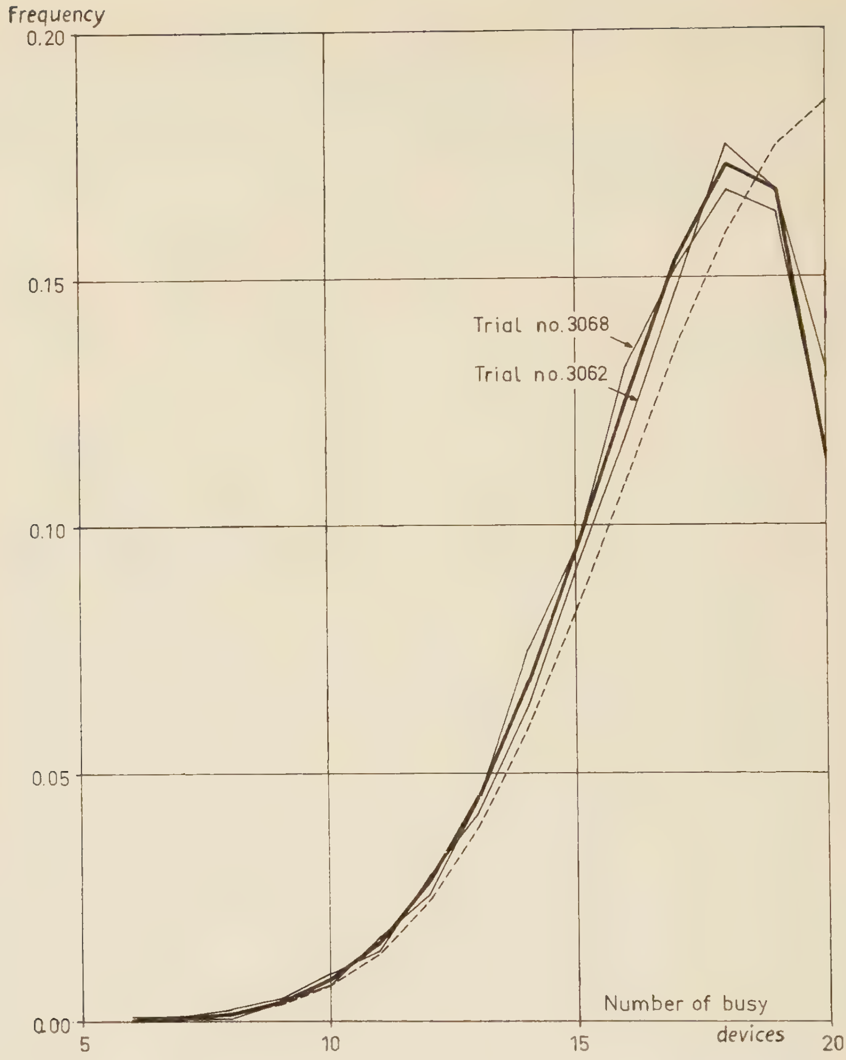

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Fig. 9. State distribution on a direct route.
$A = 21.06$, $a = 0.6$.

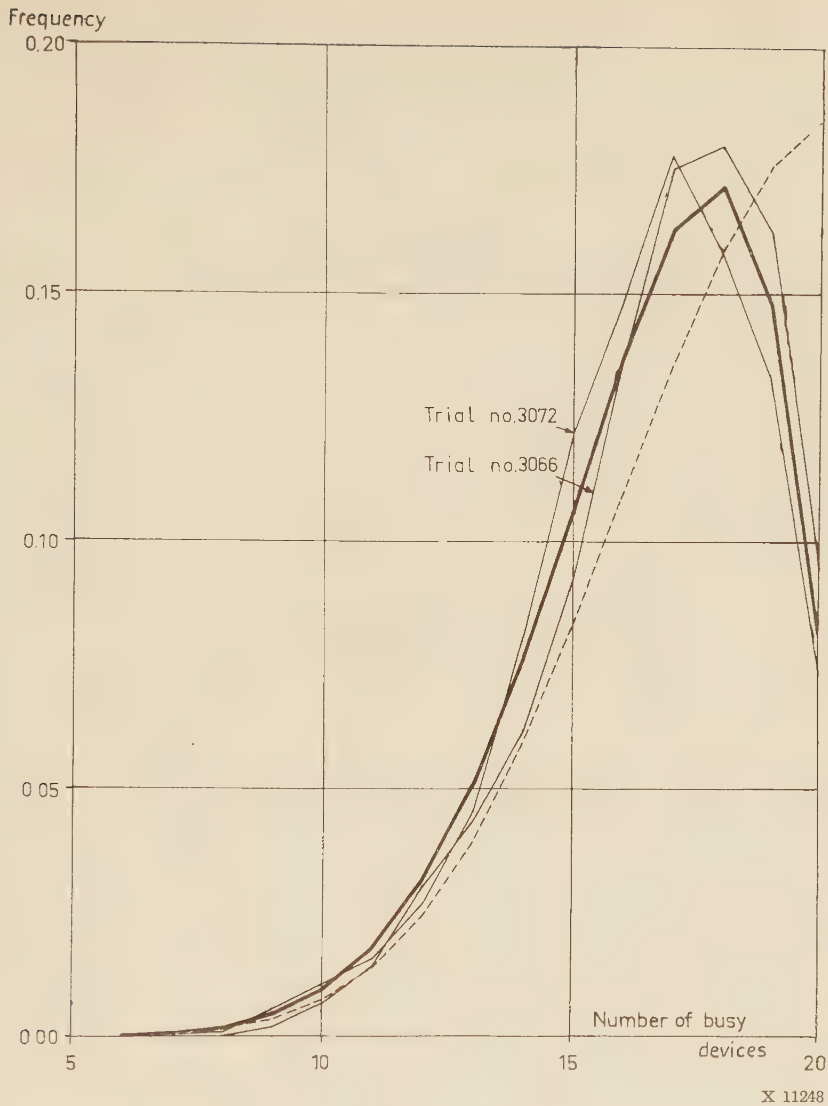

Fig. 10. State distribution on a direct route.
$A = 21.06$, $a = 0.8$.

4.3 Comparisons for alternative routes

In *Table 4* comparisons are made between calculated and measured characteristics of traffic on alternative routes. The following notations are used:

A_0 = total traffic on the alternative route calculated from

$$A_0 = v \cdot \eta \cdot n \cdot k \cdot a \cdot B + n \cdot k \cdot a \cdot (1 - v \cdot \eta) \text{ where } B \text{ is obtained from (2.2.8)}$$

A_0^* = total traffic measured on the alternative route

V_0^* = measured variance of the total traffic on the alternative route

A_{01} = traffic overflowing from the first direct route calculated from (2.3.8)

V_{01} = variance of traffic overflowing from the first direct route calculated from (2.3.9)

A_{01}^* = measured overflow traffic from the first direct route

V_{01}^* = measured variance of the overflow traffic from the first direct route

The "measured" means and variances have been obtained from the observed distributions on the alternative routes.

Table 4. Comparison with measurements on alternative routes.

$m = 20, \ n = 13, \ q = 1$

Trial no.	k	v	q_0	η	a	A_0	A_0^*	$\frac{V_0^*}{A_0^*}$	$\frac{V_{01}}{A_{01}}$	$\frac{V_{01}^*}{A_{01}^*}$
3,069	30	5	8	0.2	0.4	63.43	62.69	1.309	1.981	
3,073	30	5	8	0.2	0.4	63.43	63.91	1.225	1.981	1.724
3,070	30	5	8	0.135	0.4	71.21	71.28	0.935	—	—
3,057	27	10	1	0.1	0.4	5.09	5.01	2.004	2.343	—
3,067	20	5	8	0.2	0.6	64.19	64.01	0.784	1.952	—
3,074	20	5	8	0.2	0.6	64.19	64.62	0.722	1.952	1.746
3,068	20	5	8	0.135	0.6	72.05	71.79	0.622	—	—
3,077	20	5	8	0.08	0.6	98.64	95.01	0.620	2.039	1.544
3,058	18	10	1	0.1	0.6	5,90	5.27	1.581	2.239	—
3,065	15	5	8	0.2	0.8	65.30	64.92	0.382	1.881	—
3,075	30	10	10	0.1	0.8	130.60	129.61	0.389	1.881	1.798
3,066	15	5	8	0.135	0.8	73.31	71.87	0.361	—	—
3,072	30	10	12	0.0675	0.8	146.62	145.27	0.379	—	—
3,059	13	10	1	0.1	0.8	5.74	4.70	1.306	2.059	—

Looking first at the mean values, A_0, it is found that these agree generally very well with the measured values, A^*_0. In most cases A_0 is a little larger than A_0^*, which is in agreement with the observation that (2.2.8) generally overestimates the congestion on direct routes a little.

It is also very interesting to study the variance-to-mean ratios of the distributions. It may be recalled that this ratio is unity for a Poissonian distribution and equal to $1 - b$ for a Bernouilli distribution, b being the load per device. An overflow traffic is generally assumed to have a variance-to-mean ratio larger than unity. From *Table 4* this is seen to be true for

the values $V_{01}^*:A_{01}^*$ measured on traffic overflowing from the first direct route, but generally not for the values $V_0^*:A_0^*$ measured on the total traffic. The latter values are considerably smaller than the former, particularly when the inlet load is high. It may consequently be concluded that the total variance can be rather much smaller than the sum of the variances for each overflow traffic. However, this favourable effect can hardly be expected to prevail under real traffic conditions in a link system.

In practice the inlets are divided between several incoming routes, each deriving its traffic from a large number of subscribers. The various traffic streams in the system should therefore be practically independent, and the simple summation law for the variances in question should thus be valid.

According to the model used here, however, the inlets of the system are regarded as independent primary traffic sources with rather high calling intensities. The total traffic in the system is consequently Bernouilli-distributed with rather small values of the variance-to-mean ratio. Under these conditions the traffic streams in the system cannot be regarded as independent and the variances do not add together.

It is also seen from *Table 4* that the calculated values of $V_{01}:A_{01}$ decrease with increasing inlet load. This indicates that the variance of overflow traffic should be smaller in a link system than in a full availability system. Probably this is so, because the overflow calls should be less clustered in time in the link system. The measured values, however, show an opposite tendency. It is hard to judge whether this observation is significant or not because the accuracy of the estimates is not known.

The measured values are seen to be smaller than those calculated in every case, probably because of the limited size of the system. As a consequence the traffic offered to a direct route has been Bernouilli-distributed with a variance-to-mean ratio around 0.9 in the traffic trials, thus being more favourable than a pure Poisson traffic.

CHAPTER 5

Conclusion

In this paper the theoretically most simple type of connection in a two-stage link system with alternative routing has been considered, viz. connection to a direct route comprising a single outlet column. The theoretical study has resulted in rather simple recurrence formulas for the congestion on direct routes and for the moments of overflow traffic. They offer the means in principle of applying Wilkinson's "Equivalent Random Theory" also to link systems. The comparisons with traffic simulations have shown that the new method is very accurate for prediction of high congestion values on direct routes, while lower con-

gestion values are overestimated to some degree in systems of limited dimensions. The formulas for the variance of overflow traffic have also proved to give somewhat too high values. The simulations made up to now are not sufficiently extensive, however, to permit detailed conclusions regarding the variance estimates.

Further investigations should be made in order to decide whether the new theory is satisfactory for use in practice or not. Particularly the character of incoming traffic and its influence on the congestion should be studied. This requires measurements on real traffic as well as more extensive traffic simulations. For instance it would undoubtly be valuable to make simulations in which the inlets are divided between a number of full availability groups, each fed by a Poisson traffic. Such a model, however, requires considerably greater memory capacity than the one used here. There is indeed a great need for large and fast computers to deal with traffic simulation in the increasingly complex systems used in telephony.

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